

A PRIORI ESTIMATES AND POSITIVITY FOR SEMICLASSICAL GROUND STATES FOR SYSTEMS OF CRITICAL SCHRÖDINGER EQUATIONS IN DIMENSION TWO

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ABSTRACT. We consider in the whole plane the following Hamiltonian coupling of Schrödinger equations

$$\begin{cases} -\Delta u + V_0 u = g(v) \\ -\Delta v + V_0 v = f(u) \end{cases}$$

where $V_0 > 0$, f, g have critical growth in the sense of Moser. We prove that the (nonempty) set S of ground state solutions is compact in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ up to translations. Moreover, for each $(u, v) \in S$, one has that u, v are uniformly bounded in $L^\infty(\mathbb{R}^2)$ and uniformly decaying at infinity. Then we prove that actually the ground state is positive and radially symmetric. We apply those results to prove the existence of semiclassical ground states solutions to the singularly perturbed system

$$\begin{cases} -\varepsilon^2 \Delta \varphi + V(x) \varphi = g(\psi) \\ -\varepsilon^2 \Delta \psi + V(x) \psi = f(\varphi) \end{cases}$$

where $V \in \mathcal{C}(\mathbb{R}^2)$ is a Schrödinger potential bounded away from zero. Namely, as the adimensionalized Planck constant $\varepsilon \rightarrow 0$, we prove the existence of minimal energy solutions which concentrate around the closest local minima of the potential with some precise asymptotic rate.

1. INTRODUCTION

Consider in the whole \mathbb{R}^2 the following system of coupled Schrödinger equations

$$(1.1) \quad \begin{cases} -\varepsilon^2 \Delta \varphi + V(x) \varphi = \frac{\partial H(\varphi, \psi)}{\partial \psi} \\ -\varepsilon^2 \Delta \psi + V(x) \psi = -\frac{\partial H(\varphi, \psi)}{\partial \varphi} \end{cases}$$

where $\varepsilon > 0$, the external Schrödinger potential $V \in C(\mathbb{R}^2, \mathbb{R})$ enjoys the following condition:

$$(V) \quad 0 < V_0 := \inf_{\mathbb{R}^2} V(x) < \lim_{|x| \rightarrow \infty} V(x) = V_\infty \leq \infty.$$

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The Hamiltonian has the following form $H(\varphi, \psi) = G(\psi) - F(\varphi)$, with $F(t) = \int_0^t f(\tau) d\tau$ and $G(t) = \int_0^t g(\tau) d\tau$ and the nonlinearities $f, g \in C(\mathbb{R}, \mathbb{R})$ satisfy the following hypotheses:

- (H1) $f(t) = o(t)$ and $g(t) = o(t)$, as $t \rightarrow 0$;
- (H2) There exists $\theta > 2$ such that for any $t \neq 0$,

$$0 < \theta F(t) \leq f(t)t \text{ and } 0 < \theta G(t) \leq g(t)t;$$
- (H3) There exists $M > 0$ such that for any $t \neq 0$,

$$0 < F(t) \leq Mf(t) \text{ and } 0 < G(t) \leq Mg(t);$$
- (H4) $f(t)/|t|$ and $g(t)/|t|$ are strictly increasing for $t \neq 0$.

As a consequence of the Pohozaev-Trudinger-Moser inequality for which the Sobolev space H^1 embeds into the space of functions such that $e^{\alpha u^2} \in L^1$, the following notion of critical growth in dimension two was first introduced in [1, 14] (in the case of bounded domains):

Definition 1.1. *A function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ has critical growth in the sense of Pohozaev-Trudinger-Moser inequality, if there exists $\alpha_0 > 0$ such that*

$$\lim_{t \rightarrow +\infty} \frac{f(t)}{e^{\alpha t^2}} = \begin{cases} 0 & \text{if } \alpha > \alpha_0 \\ +\infty & \text{if } \alpha < \alpha_0 \end{cases}$$

It will be crucial in what follows the following growth assumptions:

$$(H5) \quad \liminf_{|t| \rightarrow \infty} \frac{tf(t)}{e^{\alpha_0 t^2}} \geq \beta_0 > \frac{2e}{\alpha_0} V_0 \text{ and } \liminf_{|t| \rightarrow \infty} \frac{tg(t)}{e^{\alpha_0 t^2}} \geq \beta_0 > \frac{2e}{\alpha_0} V_0.$$

It is well known, both from the theoretical point of view as well as from that of applications, that minimal energy solutions, the so-called ground states, play a fundamental role, see e.g. [3]. In what follows we will focus on this class of solutions. In particular, to investigate the sign of ground state solutions to (1.1), we require in addition the following condition:

- (H6) There exist $p, q > 1$ such that $f(t) \geq t^q$ and $g(t) \geq t^p$ for small $t > 0$;

As a reference model take $F(t) = |t|^p(e^{4\pi t^2} - 1)$ and $G(t) = |t|^q(e^{4\pi t^2} - 1)$ with $p, q > 2$ and $\alpha_0 = 4\pi$ which clearly satisfy (H1)-(H6).

Our main result reads as follows:

Theorem 1.2. *Assume condition (V) and that f, g have critical growth in the sense of Definition 1.1 and satisfy (H1)-(H5). Then, for sufficiently small $\varepsilon > 0$, (1.1) admits a least energy solution $z_\varepsilon = (\varphi_\varepsilon, \psi_\varepsilon) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$. Moreover, the following properties hold:*

- (i) *let $x_\varepsilon^1, x_\varepsilon^2, x_\varepsilon$ be any maximum point of $|\varphi_\varepsilon|, |\psi_\varepsilon|, |\varphi_\varepsilon| + |\psi_\varepsilon|$ respectively, then, setting*

$$\mathcal{M} \equiv \{x \in \mathbb{R}^2 : V(x) = V_0\}$$

one has

$$\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon, \mathcal{M}) = 0 \text{ and } \lim_{\varepsilon \rightarrow 0} |x_\varepsilon^i - x_\varepsilon| = 0, \quad i = 1, 2.$$

Furthermore, $(\varphi_\varepsilon(\varepsilon x + x_\varepsilon), \psi_\varepsilon(\varepsilon x + x_\varepsilon))$ converges (up to a subsequence) as $\varepsilon \rightarrow 0$ to a ground state solution of

$$\begin{cases} -\Delta u + V_0 u = g(v) \\ -\Delta v + V_0 v = f(u) \end{cases}$$

- (ii) if in addition (H6) holds, then replacing f and g above with their odd extensions, for $\varepsilon > 0$ small enough, up to changing sign $u_\varepsilon, v_\varepsilon > 0$ in \mathbb{R}^2 and $x_\varepsilon^1, x_\varepsilon^2$ are the unique global maximum points of $u_\varepsilon, v_\varepsilon$ respectively and which also enjoy the following

$$\lim_{\varepsilon \rightarrow 0} |x_\varepsilon^1 - x_\varepsilon^2|/\varepsilon = 0.$$

Moreover, for some $c, C > 0$ one has

$$|\varphi_\varepsilon(x)| \leq C \exp(-\frac{c}{\varepsilon}|x - x_\varepsilon^1|), \quad |\psi_\varepsilon(x)| \leq C \exp(-\frac{c}{\varepsilon}|x - x_\varepsilon^2|), \quad x \in \mathbb{R}^2;$$

(Without loss of generality, throughout the paper we may assume $0 \in \mathcal{M}$.)

Remark 1.3. Let us point out a few comments on the conditions we assume in Theorem 1.2:

- Actually the Ambrosetti-Rabinowitz condition (H2) can be replaced by the following slightly weaker assumption:

(H2)' There exists $\theta > 2$ such that for any $t \neq 0$,

$$0 < 2F(t) \leq f(t)t \text{ and } 0 < \theta G(t) \leq g(t)t,$$

or equivalently

$$0 < \theta F(t) \leq f(t)t \text{ and } 0 < 2G(t) \leq g(t)t.$$

- We also point out that conditions (H2) and (H4) are weaker than the following assumption:

(H) $f, g \in C^1(\mathbb{R}, \mathbb{R})$ and there exists $\delta' > 0$ such that for any $s \neq 0$,

$$0 < (1 + \delta')f(s)s \leq f'(s)s^2 \text{ and } 0 < (1 + \delta')g(s)s \leq g'(s)s^2$$

which appears in the literature, see [4, 27, 29].

- Hypotheses (H6) and (H) can be also found in [4]. Clearly hypothesis (H) yields $sf(s) \leq f(1)|s|^{2+\delta'}$ and $sg(s) \leq g(1)|s|^{2+\delta'}$ if $|s| \leq 1$. Let us point out that in the present paper we do not require $sf(s), sg(s)$ to be less than $|s|^r$ near the origin for some $r > 2$.

Systems of the form (1.1) have been largely investigated in the last three decades being a prototype in many different applications, where they model for instance the minimal energy interaction between nonlinear fields, see [3, 36]. The scenario changes remarkably from the higher dimensional case $N \geq 3$ to the planar case $N = 2$. In particular, $N = 2$ affects the notion of critical growth which is the maximal admissible growth for the nonlinearities in order to preserve the variational structure of the problem; we refer to [8–10] for a discussion on this topic and to [5, 30] for a survey on systems of the form (1.1) in the case of bounded domains. As far as we are concerned with minimal energy solutions in the whole space, existence results have been first established in [31], see also [34], in the higher dimensional case and then recently extended to $N = 2$ in [16], where the Trudinger-Moser

critical case is covered, see also [4, 6]. Qualitative properties of minimal energy solutions such as symmetry and positivity have been investigated in the higher dimensional case in [4, 7, 32], see also [12, 28] for closely related results. Always in dimension $N \geq 3$, a priori estimates have been obtained in [15]. A priori bounds open the way to investigate the existence and concentrating behavior, as $\varepsilon \rightarrow 0$, of the so-called semiclassical states. From the point of view of Physics, these solutions live on the interface between quantum and classical Mechanics, in the sense that the field behaves like a Newtonian particle as $\varepsilon \rightarrow 0$, see [21] for a survey on the topic and references therein. Semiclassical states for singularly perturbed Schrödinger systems have been studied in the higher dimensional case in [2, 17, 29].

Finally, let us mention that the question whether the ground state we find is unique, seems to be out of reach at the moment. This is still a challenging open problem even in the subcritical case as well as in higher dimensions in which uniqueness of positive solutions (not necessarily ground states) is known just in a few particular cases such as Lane-Emden systems [11]. More in general, the matter of uniqueness of ground states, even in cases in which one has multiplicity of positive solutions, remains open even for the single equation.

Overview. The paper is organized as follows: in Section 2 we begin with studying a limit problem for system (1.1). Here we complete the work initiated in [16], where the existence of ground states is proved, by establishing a priori estimates, regularity, symmetry and qualitative properties of solutions. Here we use a suitable Nehari manifold approach in the spirit of Pankov [25] combined with Moser type techniques, as everything is set in dimension two and in presence of Moser critical growth. In particular we exploit those preliminary results to prove positivity of ground states solutions in a quite general setting, as developed throughout Section 2.4. Then, Section 3 is devoted to apply the informations previously obtained on the limit problem, to analyze the concentrating behavior of semiclassical solutions from the point of view of localizing bumps as well as of deriving the asymptotic rate of concentration. Here the presence of critical Moser's growth requires some delicate energy estimates which we then apply to establish compactness.

2. THE LIMIT PROBLEM

By denoting $u_\varepsilon(x) = \varphi(\varepsilon x)$, $v_\varepsilon(x) = \psi(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$, (1.1) is equivalent to

$$\begin{cases} -\Delta u_\varepsilon + V_\varepsilon(x)u_\varepsilon = g(v_\varepsilon) \\ -\Delta v_\varepsilon + V_\varepsilon(x)v_\varepsilon = f(u_\varepsilon) \end{cases}$$

in the whole plane. Let $x_0 \in \mathbb{R}^2$ and assume $u_\varepsilon(\cdot + \frac{x_0}{\varepsilon}) \rightarrow u$, $v_\varepsilon(\cdot + \frac{x_0}{\varepsilon}) \rightarrow v$ in $C_{loc}^1(\mathbb{R}^2)$, if $V_0 = V(x_0)$ then one has

$$(2.1) \quad \begin{cases} -\Delta u + V_0 u = g(v) \\ -\Delta v + V_0 v = f(u) \end{cases}$$

which is the so-called limit problem of (1.1). Recently, D. G. De Figueiredo, J. M. do Ó and J. Zhang established in [16] the existence of ground state solutions to (2.1), precisely

Theorem A. (Theorem 1.3 in [16]) *Suppose that f, g have critical growth and satisfy (H1)–(H5). Then (2.1) admits a ground state solution $(u, v) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$.*

Denote by \mathcal{S} the set of ground state solutions to system (2.1), then by Theorem A $\mathcal{S} \neq \emptyset$. Here we investigate the regularity and qualitative properties of the ground state solutions to (2.1). Precisely, we prove the following results:

Theorem 2.1. *Suppose f, g have critical growth and satisfy (H1)–(H5). Then the following hold true:*

- (i) $(u, v) \in \mathcal{S} \Rightarrow u, v \in L^\infty(\mathbb{R}^2) \cap C_{loc}^{1,\gamma}(\mathbb{R}^2)$ for some $\gamma \in (0, 1)$;
- (ii) let $x_z \in \mathbb{R}^2$ be the maximum point of $|u(x)| + |v(x)|$, then the set

$$\{(u(\cdot + x_z), v(\cdot + x_z)) \mid (u, v) \in \mathcal{S}\}$$

is compact in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$;

- (iii) $0 < \inf\{\|u\|_\infty, \|v\|_\infty \mid (u, v) \in \mathcal{S}\} \leq \sup\{\|u\|_\infty, \|v\|_\infty \mid (u, v) \in \mathcal{S}\} < \infty$;
- (iv) $u(x + x_z) \rightarrow 0$ and $v(x + x_z) \rightarrow 0$, as $|x| \rightarrow \infty$ uniformly for any $z = (u, v) \in \mathcal{S}$, where x_z is given in (ii);
- (v) for any $(u, v) \in \mathcal{S}$, the following Pohozaev-type identity holds

$$\int_{\mathbb{R}^2} (F(u) + G(v) - V_0 uv) dx = 0.$$

Theorem 2.2. *Assume in addition to the hypotheses of Theorem 2.1 that also (H6) holds. Then, replacing f and g in Theorem 2.1 with their odd extensions, for any $(u, v) \in \mathcal{S}$ one has $u, v \in C^2(\mathbb{R}^2)$ and $uv > 0$ in \mathbb{R}^2 . Moreover, there exists some point $x_0 \in \mathbb{R}^2$ such that u, v are radially symmetric with respect to the same point x_0 , namely $u(x) = u(|x - x_0|)$, $v(x) = v(|x - x_0|)$ and setting $r = |x - x_0|$, one has for $r > 0$*

$$\frac{\partial u}{\partial r} < 0 \quad \text{and} \quad \frac{\partial v}{\partial r} < 0$$

as well as

$$\Delta u(x_0) < 0 \quad \text{and} \quad \Delta v(x_0) < 0.$$

Moreover, there exist $C, c > 0$, independent of $z = (u, v) \in \mathcal{S}$, such that

$$|D^\alpha u(x)| + |D^\alpha v(x)| \leq C \exp(-c|x - x_0|), \quad x \in \mathbb{R}^2, |\alpha| = 0, 1$$

2.1. The functional setting: a generalized Nehari manifold. Let $H^1(\mathbb{R}^2)$ be the usual Sobolev space endowed with the inner product

$$(u, v)_{H^1} := \int_{\mathbb{R}^2} \nabla u \nabla v + V_0 uv, \quad \|u\|_{H^1}^2 := (u, u)_{H^1}, \quad u, v \in H^1(\mathbb{R}^2).$$

and set $E = H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ with the inner product

$$(z_1, z_2) := (u_1, u_2)_{H^1} + (v_1, v_2)_{H^1}, \quad z_i = (u_i, v_i) \in E, i = 1, 2.$$

Clearly we have the space decomposition $E = E^+ \oplus E^-$, where

$$E^+ := \{(u, u) \mid u \in H^1(\mathbb{R}^2)\} \quad \text{and} \quad E^- := \{(u, -u) \mid u \in H^1(\mathbb{R}^2)\}.$$

For each $z = (u, v) \in E$, one has

$$z = z^+ + z^- = ((u + v)/2, (u + v)/2) + ((u - v)/2, (v - u)/2).$$

Weak solutions to (2.1) are the critical points of the associated energy functional

$$\Phi(z) := \int_{\mathbb{R}^2} \nabla u \nabla v + V_0 uv - I(z), \quad z = (u, v) \in E,$$

where $I(z) = \int_{\mathbb{R}^2} F(u) + G(v)$. Using the above notation we have

$$(2.2) \quad \Phi(z) := \frac{1}{2} \|z^+\|^2 - \frac{1}{2} \|z^-\|^2 - I(z),$$

which emphasizes the strongly indefinite nature of Φ which however, by the hypotheses on f and g , is of class $C^1(E, \mathbb{R})$ and

$$(2.3) \quad I(0) = 0, \quad \langle I'(z), z \rangle > 2I(z) > 0, \quad \text{for all } z \in E \setminus \{0\}.$$

On one hand, if $z = (u, v) \in E \setminus \{0\}$ such that $\Phi'(z) = 0$, then by (H2)

$$\Phi(z) = \Phi(z) - \frac{1}{2} \langle \Phi'(z), z \rangle = \int_{\mathbb{R}^2} \frac{1}{2} f(u)u - F(u) + \frac{1}{2} g(v)v - G(v) > 0.$$

On the other hand, if $z = (u, -u) \in E^-$, we have by (H2)

$$\Phi(z) = - \int_{\mathbb{R}^2} (|\nabla u|^2 + V_0 u^2) - \int_{\mathbb{R}^2} F(u) + G(-u) \leq 0.$$

As a consequence, if $z \in E$ is a nontrivial critical point of Φ , then necessarily $z \in E \setminus E^-$. This motivates the introduction of the following generalized Nehari manifold, due to Pankov [25] and then used also in [16, 33, 34]:

$$\mathcal{N} := \{z \in E \setminus E^- : \langle \Phi'(z), z \rangle = 0, \langle \Phi'(z), \varphi \rangle = 0 \text{ for all } \varphi \in E^-\}.$$

Let

$$c_* := \inf_{z \in \mathcal{N}} \Phi(z)$$

then c_* is called the least energy level of system (2.1). In [16] the authors proved that $c_* \in (0, 4\pi/\alpha_0)$ and that it is achieved on \mathcal{N} .

2.2. Proof of Theorem 2.1. Let $\{z_n\} \subset S$, namely

$$(3.4) \quad \Phi(z_n) = c_* \quad \text{and} \quad \Phi'(z_n) = 0, \quad \forall n \in \mathbb{N}$$

We carry out the proof of (ii) of Theorem 2.1 through the following four steps:

- We first prove that $\{z_n\}$ is bounded in E (Proposition 2.3);
- In Proposition 2.4 we prove that there exist $\{y_n\} \subset \mathbb{R}^2$ and $z_0 \neq \mathbf{0}$ such that $z_n(\cdot + y_n) \rightharpoonup z_0$ in E and $z_n(\cdot + y_n) \xrightarrow{a.e.} z_0$ in \mathbb{R}^2 , as $n \rightarrow \infty$;
- In Proposition 2.5 we show that z_0 is actually a critical point of Φ ;
- Finally in Proposition 2.6 we prove that $z_0 \in S$ and that actually $z_n(\cdot + y_n) \rightarrow z_0$ strongly in E , as $n \rightarrow \infty$.

In the proof of the Proposition 2.3 below we will use the following lemma which we borrow from [13]:

Lemma A. *The following inequality holds*

$$s \leq \begin{cases} (e^{t^2} - 1) + s(\log s)^{1/2}, & \text{for all } t \geq 0 \text{ and } s \geq e^{1/4}; \\ (e^{t^2} - 1) + \frac{1}{2}s^2, & \text{for all } t \geq 0 \text{ and } 0 \leq s \leq e^{1/4}. \end{cases}$$

The proofs of Proposition 2.3 and 2.5 are similar to [16], however for the sake of completeness we give the details.

Proposition 2.3. *There exists $C > 0$ such that for all $n \in \mathbb{N}$:*

- 1) $\|z_n\| = \|(u_n, v_n)\| \leq C$;
- 2) $\int_{\mathbb{R}^2} f(u_n)u_n \, dx \leq C$ and $\int_{\mathbb{R}^2} g(v_n)v_n \, dx \leq C$;
- 3) $\int_{\mathbb{R}^2} F(u_n) \, dx \leq C$ and $\int_{\mathbb{R}^2} G(v_n) \, dx \leq C$.

Proof. From $\langle \Phi'(z_n), z_n \rangle = 0$ we have

$$(3.5) \quad 2 \int_{\mathbb{R}^2} (\nabla u_n \nabla v_n + V_0 u_n v_n) \, dx - \int_{\mathbb{R}^2} f(u_n)u_n \, dx - \int_{\mathbb{R}^2} g(v_n)v_n \, dx = 0.$$

Recalling that

$$\Phi(z_n) = \int_{\mathbb{R}^2} (\nabla u_n \nabla v_n + V_0 u_n v_n) \, dx - \int_{\mathbb{R}^2} (F(u_n) + G(v_n)) \, dx = c_*$$

we obtain by (H_3) the following

$$\begin{aligned} \int_{\mathbb{R}^2} [f(u_n)u_n + g(v_n)v_n] \, dx &= 2 \int_{\mathbb{R}^2} [F(u_n) + G(v_n)] \, dx + 2c_* \\ &\leq \frac{2}{\theta} \int_{\mathbb{R}^2} [f(u_n)u_n + g(v_n)v_n] \, dx + 2c_*. \end{aligned}$$

Thus

$$(3.6) \quad \int_{\mathbb{R}^2} [f(u_n)u_n + g(v_n)v_n] \, dx \leq \frac{2c_*\theta}{\theta - 2}.$$

From $\langle \Phi'(z_n), (v_n, 0) \rangle = 0$ and $\langle \Phi'(z_n), (0, u_n) \rangle = 0$, we have

$$\|v_n\|^2 - \int_{\mathbb{R}^2} f(u_n)v_n \, dx = 0 \quad \text{and} \quad \|u_n\|^2 - \int_{\mathbb{R}^2} g(v_n)u_n \, dx = 0.$$

Let $U_n = u_n/\|u_n\|$ and $V_n = v_n/\|v_n\|$, then

$$(3.7) \quad \|v_n\| = \int_{\mathbb{R}^2} f(u_n)V_n \, dx,$$

$$(3.8) \quad \|u_n\| = \int_{\mathbb{R}^2} g(v_n)U_n \, dx.$$

By (H_1) , there exist $\beta > 0$ and $C_\beta > 0$ such that

$$f(t) \leq C_\beta e^{\beta t^2} \quad \text{and} \quad g(t) \leq C_\beta e^{\beta t^2} \quad \text{for all } t \geq 0.$$

Moreover, there exists $C_1 > 0$ such that for all n

$$f(u_n(x)) \leq C_1 u_n(x) \quad \text{for } x \in \{\mathbb{R}^2 : f(u_n(x))/C_\beta \leq e^{1/4}\}.$$

Setting $t = V_n$ and $s = f(u_n)/C_\beta$ in Lemma A then by (H1)-(H2) together with the Pohozaev-Trudinger-Moser inequality, we get

$$\begin{aligned} \int_{\mathbb{R}^2} f(u_n) V_n \, dx &\leq C_\beta \int_{\{x \in \mathbb{R}^2 : f(u_n(x))/C_\beta \geq e^{1/4}\}} \frac{1}{C_\beta} f(u_n) \left[\log\left(\frac{1}{C_\beta} f(u_n)\right) \right]^{1/2} dx \\ &\quad + \frac{1}{2} \int_{\{x \in \mathbb{R}^2 : f(u_n(x))/C_\beta \leq e^{1/4}\}} \frac{1}{C_\beta} (f(u_n))^2 \, dx + C_\beta \int_{\mathbb{R}^2} (e^{V_n^2} - 1) \, dx \\ &\leq C_2 + (\beta^{1/2} + C_1/(2C_\beta)) \int_{\mathbb{R}^2} f(u_n) u_n \, dx, \end{aligned}$$

for some constant $C_2 > 0$. This estimate together with (3.7) implies, for some constant $c_1 > 0$, that

$$(3.9) \quad \|v_n\| \leq c_1 \left(1 + \int_{\mathbb{R}^2} f(u_n) u_n \, dx\right)$$

and similarly

$$(3.10) \quad \|u_n\| \leq c_1 \left(1 + \int_{\mathbb{R}^2} g(v_n) v_n \, dx\right).$$

From (3.9), (3.10) and (3.6) it follows the first claim 1). Then, by (3.6) and (H₃) we obtain the remaining bounds 2) and 3). \square

Next we prove that, up to translations, $\{z_n\}$ has a nontrivial weak limit. Clearly (u_n, v_n) satisfies just one of the following conditions:

(*Vanishing*) $\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^2} \int_{B_R(y)} (u_n^2 + v_n^2) \, dx = 0$ for all $R > 0$;

(*Nonvanishing*) there exist $\nu > 0$, $R_0 > 0$ and $\{y_n\} \subset \mathbb{R}^2$ such that

$$\lim_{n \rightarrow \infty} \int_{B_{R_0}(y_n)} (u_n^2 + v_n^2) \, dx \geq \nu.$$

We borrow from [13] the following lemma:

Lemma B. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $f \in C(\mathbb{R}, \mathbb{R})$. Let $\{u_n\} \subset L^1(\Omega)$ be such that $u_n \rightarrow u$ strongly in $L^1(\Omega)$,*

$$f(u_n) \in L^1(\Omega) \quad \text{and} \quad \int_{\Omega} |f(u_n) u_n| \, dx \leq C, n \geq 1$$

for some $C > 0$. Then, up to a subsequence we have

$$f(u_n) \rightarrow f(u) \quad \text{strongly in } L^1(\Omega) \quad \text{as } n \rightarrow \infty.$$

Proposition 2.4. *Vanishing does not occur.*

Proof. We know from [16] that $c_* \in (0, 4\pi/\alpha_0)$, hence for some $\delta > 0$ sufficiently small one has $c_* \in (0, 4\pi/\alpha_0 - \delta)$. Assume by contradiction that vanishing occurs, namely

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^2} \int_{B_R(y)} (u_n^2 + v_n^2) \, dx = 0 \quad \text{for all } R > 0,$$

then Lions's lemma [23] yields $u_n \rightarrow 0, v_n \rightarrow 0$ strongly in $L^s(\mathbb{R}^2)$ for any $s > 2$.

Let us divide the proof into two steps:

Step 1. We claim

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} F(u_n) dx = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} G(v_n) dx = 0.$$

Indeed, by Lemma B, for any $R > 0$ one has $f(u_n) \rightarrow 0$ and $g(v_n) \rightarrow 0$ strongly in $L^1(B_R(0))$ as $n \rightarrow \infty$. Then by (H3) and the Lebesgue dominated convergence theorem,

$$(3.11) \quad \lim_{n \rightarrow \infty} \int_{B_R(0)} F(u_n) dx = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{B_R(0)} G(v_n) dx = 0.$$

In order to prove the claim, it is enough to prove that for any $\varepsilon > 0$, there exists $R > 0$ such that for n large enough,

$$(3.12) \quad \int_{\mathbb{R}^2 \setminus B_R(0)} F(u_n) dx \leq \varepsilon \quad \text{and} \quad \int_{\mathbb{R}^2 \setminus B_R(0)} G(v_n) dx \leq \varepsilon.$$

By (H3) and Proposition 2.3, for any $K > 0$ and n ,

$$\int_{\{x \in \mathbb{R}^2 \setminus B_R(0) : |u_n(x)| \geq K\}} F(u_n) \leq \frac{M}{K} \int_{\{x \in \mathbb{R}^2 \setminus B_R(0) : |u_n(x)| \geq K\}} f(u_n) u_n \leq \frac{MC}{K}.$$

Then choosing $K > 0$ large enough, we get that for all n

$$(3.13) \quad \int_{\{x \in \mathbb{R}^2 \setminus B_R(0) : |u_n(x)| \geq K\}} F(u_n) \leq \frac{\varepsilon}{2}.$$

By (H1), for any $\rho > 0$ there exists $C_{\rho,K} > 0$ such that

$$F(t) \leq \rho t^2 + C_{\rho,K} t^4, \quad |t| \leq K.$$

Recalling that $u_n \rightarrow 0$ strongly in $L^4(\mathbb{R}^2)$, we obtain

$$\limsup_{n \rightarrow \infty} \int_{\{x \in \mathbb{R}^2 \setminus B_R(0) : |u_n(x)| \leq K\}} F(u_n) \leq \rho \sup_n \|u_n\|_2^2.$$

By Proposition 2.3 and since ρ is arbitrary, for n large enough we get

$$(3.14) \quad \int_{\{x \in \mathbb{R}^2 \setminus B_R(0) : |u_n(x)| \leq K\}} F(u_n) \leq \frac{\varepsilon}{2}.$$

Thus (3.13) and (3.14) yield the first bound in (3.12) and similarly one gets the second bound.

Step 2. We claim that $c_* = 0$, from which the contradiction follows as we know $c_* > 0$. We need the following inequality used in [18, Lemma 4.1]

$$(3.15) \quad t s \leq t^2(e^{t^2} - 1) + s(\log s)^{\frac{1}{2}}, \quad \text{for all } (t, s) \in [0, \infty) \times [e^{\frac{1}{34}}, \infty).$$

By Step 1,

$$(3.16) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} (\nabla u_n \nabla v_n + V_0 u_n v_n) = c_*.$$

If $u_n \rightarrow 0$ or $v_n \rightarrow 0$ strongly in $H^1(\mathbb{R}^2)$ as $n \rightarrow \infty$, then (3.16) directly yields $c_* = 0$. Therefore, let us assume $\inf_{n \geq 1} \|u_n\| \geq b > 0$. Note that

$$(3.17) \quad \|u_n\|^2 = \int_{\mathbb{R}^2} g(v_n) u_n \, dx.$$

By (H1), for any fixed $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$f(t), g(t) \leq C_\varepsilon e^{(\alpha_0 + \varepsilon)t^2} \quad \text{for } t \geq 0.$$

Let $\bar{u}_n = (4\pi/\alpha_0 - \delta)^{1/2} u_n / \|u_n\|$ and using the inequality (3.15) with $s = g(v_n)/C_\varepsilon$ and $t = \sqrt{\alpha_0} |\bar{u}_n|$,

$$\begin{aligned} (4\pi/\alpha_0 - \delta)^{1/2} \|u_n\| &\leq \int_{\mathbb{R}^2} g(v_n) |\bar{u}_n| \, dx \\ &= \frac{C_\varepsilon}{\sqrt{\alpha_0}} \int_{\mathbb{R}^2} \frac{g(v_n)}{C_\varepsilon} \sqrt{\alpha_0} |\bar{u}_n| \, dx \\ &\leq \frac{C_\varepsilon}{\sqrt{\alpha_0}} \int_{\{x \in \mathbb{R}^2 : g(v_n(x))/C_\varepsilon \geq e^{1/\sqrt[3]{4}}\}} \frac{g(v_n)}{C_\varepsilon} [\log(\frac{g(v_n)}{C_\varepsilon})]^{1/2} \, dx \\ &\quad + \int_{\{x \in \mathbb{R}^2 : g(v_n(x))/C_\varepsilon \leq e^{1/\sqrt[3]{4}}\}} g(v_n) |\bar{u}_n| \, dx + C_\varepsilon \sqrt{\alpha_0} \int_{\mathbb{R}^2} \bar{u}_n^2 (e^{\alpha_0 \bar{u}_n^2} - 1) \, dx \\ &\leq \sqrt{\frac{\alpha_0 + \varepsilon}{\alpha_0}} \int_{\{x \in \mathbb{R}^2 : g(v_n(x))/C_\varepsilon \geq e^{1/\sqrt[3]{4}}\}} g(v_n) v_n \, dx + C_\varepsilon \sqrt{\alpha_0} \int_{\mathbb{R}^2} \bar{u}_n^2 (e^{\alpha_0 \bar{u}_n^2} - 1) \, dx \\ &\quad + \int_{\{x \in \mathbb{R}^2 : g(v_n(x))/C_\varepsilon \leq e^{1/\sqrt[3]{4}}\}} g(v_n) |\bar{u}_n| \, dx \\ &\leq \sqrt{\frac{\alpha_0 + \varepsilon}{\alpha_0}} \int_{\mathbb{R}^2} g(v_n) v_n \, dx + I_{1,n} + I_{2,n}. \end{aligned}$$

Recalling that $\bar{u}_n \rightarrow 0$ strongly in $L^s(\mathbb{R}^2)$ for any $s > 2$. Since $\|\bar{u}_n\|^2 = 4\pi/\alpha_0 - \delta$, there exists $p > 1$ (close to 1) such that $p\alpha_0(4\pi/\alpha_0 - \delta) < 4\pi$. Thus, by the Pohozaev-Trudinger-Moser inequality, as $n \rightarrow \infty$,

$$I_{1,n} \leq C_\varepsilon \sqrt{\alpha_0} \left(\int_{\mathbb{R}^2} |\bar{u}_n|^{2q} \right)^{1/q} \left(\int_{\mathbb{R}^2} (e^{p\alpha_0 \bar{u}_n^2} - 1) \right)^{1/p} \rightarrow 0,$$

where $1/p + 1/q = 1$, namely, $I_{1,n} = o_n(1)$. Note that by (H1)-(H2), for any $\rho > 0$, there exists $C_{\rho,\varepsilon} > 0$ such that

$$g(v_n(x)) \leq \rho |v_n(x)| + C_{\rho,\varepsilon} v_n^2, \quad \text{for any } x \in \mathbb{R}^2 \text{ with } g(v_n(x))/C_\varepsilon \leq e^{1/\sqrt[3]{4}}.$$

Then

$$I_{2,n} \leq \int_{\mathbb{R}^2} (\rho |v_n| + C_{\rho,\varepsilon} v_n^2) |\bar{u}_n| \, dx \leq \left[\rho \left(\int_{\mathbb{R}^2} |v_n|^2 \right)^{1/2} + C_{\rho,\varepsilon} \left(\int_{\mathbb{R}^2} |v_n|^4 \right)^{1/2} \right] \left(\int_{\mathbb{R}^2} |\bar{u}_n|^2 \right)^{1/2}.$$

Recalling $v_n \rightarrow 0$ strongly in $L^4(\mathbb{R}^2)$,

$$\limsup_{n \rightarrow \infty} I_{2,n} \leq C' \rho,$$

where $C' > 0$ is independent of ρ . By the arbitrary choice of ρ , $I_{2,n} = o_n(1)$. Hence,

$$(3.18) \quad (4\pi/\alpha_0 - \delta)^{1/2} \|u_n\| \leq o_n(1) + (1 + \frac{\varepsilon}{\alpha_0})^{1/2} \int_{\mathbb{R}^2} g(v_n) v_n.$$

Similarly, we have

$$(3.19) \quad (4\pi/\alpha_0 - \delta)^{1/2} \|v_n\| \leq o_n(1) + (1 + \frac{\varepsilon}{\alpha_0})^{1/2} \int_{\mathbb{R}^2} f(u_n) u_n.$$

Note that

$$\langle \Phi'(z_n), z_n \rangle = 2 \int_{\mathbb{R}^2} (\nabla u_n \nabla v_n + V_0 u_n v_n) - \int_{\mathbb{R}^2} f(u_n) u_n + \int_{\mathbb{R}^2} g(v_n) v_n = 0$$

and that by (3.16) we get

$$\int_{\mathbb{R}^2} f(u_n) u_n + \int_{\mathbb{R}^2} g(v_n) v_n = 2c_* + o_n(1).$$

It follows from (3.18)-(3.19) that

$$(4\pi/\alpha_0 - \delta)^{1/2} (\|u_n\|_{H^1} + \|v_n\|_{H^1}) \leq 2c_* (1 + \frac{\varepsilon}{\alpha_0})^{1/2} + o_n(1).$$

Since $c_* < 4\pi/\alpha_0 - \delta$, for $\varepsilon > 0$ sufficiently small, as n is large enough we have

$$\|u_n\|_{H^1} + \|v_n\|_{H^1} \leq 2(4\pi/\alpha_0 - \delta/2)^{1/2}.$$

Then similarly as above, by the Trudinger-Moser inequality and $u_n \rightarrow 0$ strongly in $L^q(\mathbb{R}^2)$ for any $q > 2$, we have $\int_{\mathbb{R}^2} g(v_n) u_n \rightarrow 0$, which implies by (3.17) that $u_n \rightarrow 0$ strongly in $H^1(\mathbb{R}^2)$. Thus, it follows from (3.16) that $c_* = 0$ and hence a contradiction and vanishing does not occur. \square

As a consequence of Proposition 2.4, up to a subsequence, there exist $\{y_n\} \subset \mathbb{R}^2$ and $z_0 \neq 0$ such that $z_n(\cdot + y_n) \rightharpoonup z_0$ in E and $z_n(\cdot + y_n) \xrightarrow{a.e.} z_0$ in \mathbb{R}^2 , as $n \rightarrow \infty$.

Proposition 2.5. *The weak limit z_0 is a critical point of Φ .*

Proof. By (H1), there exist $a > 0$ and $\alpha > \alpha_0$ such that

$$|f(t)| \leq a|t| + (e^{\alpha t^2} - 1) \quad \text{for all } t \in \mathbb{R}.$$

Then by the Pohozaev-Trudinger-Moser inequality $f(\bar{u}_n) \in L_{loc}^1(\mathbb{R}^2)$ and $g(\bar{v}_n) \in L_{loc}^1(\mathbb{R}^2)$, where $\bar{z}_n = (\bar{u}_n, \bar{v}_n) = (u(\cdot + y_n), v(\cdot + y_n))$. From Lemma B and Proposition 2.3 we get, as $n \rightarrow \infty$

$$\int_{\mathbb{R}^2} (f(\bar{u}_n) \varphi + g(\bar{v}_n) \phi) \rightarrow \int_{\mathbb{R}^2} (f(u_0) \varphi + g(v_0) \phi),$$

for any $(\varphi, \phi) \in C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)$. Noting that $\Phi'(\bar{z}_n) = 0$, it follows that

$$\int_{\mathbb{R}^2} (\nabla u_0 \nabla \phi + \nabla v_0 \nabla \varphi + V_0 u_0 \phi + V_0 v_0 \varphi) dx = \int_{\mathbb{R}^2} (f(u_0) \varphi + g(v_0) \phi) dx,$$

for any $(\varphi, \phi) \in C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)$. Thus, $\Phi'(z_0) = 0$ in E and $z_0 = (u_0, v_0)$ is a critical point of Φ . \square

Proposition 2.6. $z_0 \in \mathcal{S}$ and $z_n(\cdot + y_n) \rightarrow z_0$ in E , as $n \rightarrow \infty$, thus \mathcal{S} is a compact set.

Proof. Thanks to the invariance of Φ by translation, let us write for simplicity z_n in place of $z_n(\cdot + y_n)$ and let $z_n = (u_n, v_n)$, $z_0 = (u_0, v_0)$. By (H2), $f(s)s - 2F(s) \geq 0$ and $g(s)s - 2G(s) \geq 0$ for any $s \in \mathbb{R}$. Then by Fatou's Lemma,

$$\begin{aligned}
 c_* &= \Phi(z_n) - \frac{1}{2} \langle \Phi'(z_n), z_n \rangle \\
 &= \lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^2} \frac{1}{2} f(u_n) u_n - F(u_n) + \int_{\mathbb{R}^2} \frac{1}{2} g(v_n) v_n - G(v_n) \right) \\
 &\geq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^2} \frac{1}{2} f(u_n) u_n - F(u_n) + \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} \frac{1}{2} g(v_n) v_n - G(v_n) \\
 (3.20) \quad &\geq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} \frac{1}{2} f(u_n) u_n - F(u_n) + \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} \frac{1}{2} g(v_n) v_n - G(v_n) \\
 &\geq \int_{\mathbb{R}^2} \frac{1}{2} f(u_0) u_0 - F(u_0) + \int_{\mathbb{R}^2} \frac{1}{2} g(v_0) v_0 - G(v_0) \\
 &= \Phi(z_0) - \frac{1}{2} \langle \Phi'(z_0), z_0 \rangle = \Phi(z_0).
 \end{aligned}$$

On the other hand, since $z_0 \neq 0$ and $\Phi'(z_0) = 0$ one has $\Phi(z_0) \geq c_*$. Therefore, z_0 is a ground state solution of (2.1), namely, $z_0 \in \mathcal{S}$.

Next we prove that $z_n \rightarrow z_0$ in E . By (3.20) and $\Phi(z_0) = c_*$ we have

$$(3.21) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \frac{1}{2} f(u_n) u_n - F(u_n) = \int_{\mathbb{R}^2} \frac{1}{2} f(u_0) u_0 - F(u_0)$$

and

$$(3.22) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \frac{1}{2} g(v_n) v_n - G(v_n) = \int_{\mathbb{R}^2} \frac{1}{2} g(v_0) v_0 - G(v_0).$$

By (H2) we get

$$0 \leq \frac{\theta - 2}{2} F(u_n) \leq \frac{1}{2} f(u_n) u_n - F(u_n), \quad 0 \leq \frac{\theta - 2}{2} G(v_n) \leq \frac{1}{2} g(v_n) v_n - G(v_n), \quad n \geq 1$$

and the Lebesgue dominated convergence theorem, together with (3.21) and (3.22) yields

$$(3.23) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} F(u_n) = \int_{\mathbb{R}^2} F(u_0), \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} G(v_n) = \int_{\mathbb{R}^2} G(v_0).$$

Then, by (3.21) and (3.22) one has

$$(3.24) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} f(u_n) u_n = \int_{\mathbb{R}^2} f(u_0) u_0, \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} g(v_n) v_n = \int_{\mathbb{R}^2} g(v_0) v_0.$$

Since $z_n, z_0 \in S$, we have

$$\int_{\mathbb{R}^2} \nabla u_n \nabla v_n + V_0 u_n v_n = c_* + \int_{\mathbb{R}^2} F(u_n) + G(v_n),$$

$$\int_{\mathbb{R}^2} \nabla u_0 \nabla v_0 + V_0 u_0 v_0 = c_* + \int_{\mathbb{R}^2} F(u_0) + G(v_0).$$

Thanks to (3.23),

$$(3.25) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \nabla u_n \nabla v_n + V_0 u_n v_n = \int_{\mathbb{R}^2} \nabla u_0 \nabla v_0 + V_0 u_0 v_0.$$

By $\langle \Phi'(u_n, v_n), (u_n, u_n) \rangle = 0$ and (3.24)-(3.25) we obtain

$$(3.26) \quad \int_{\mathbb{R}^2} |\nabla u_n|^2 + V_0 u_n^2 = \int_{\mathbb{R}^2} f(u_0) u_0 + g(v_n) u_n - \int_{\mathbb{R}^2} \nabla u_0 \nabla v_0 + V_0 u_0 v_0 + o_n(1).$$

At the same time from $\langle \Phi'(u_n, v_n), (u_n, -u_n) \rangle = 0$ and $\langle \Phi'(u_0, v_0), (u_0, -u_0) \rangle = 0$, we have

$$(3.27) \quad \int_{\mathbb{R}^2} f(u_n) u_n = \int_{\mathbb{R}^2} g(v_n) u_n, \quad \int_{\mathbb{R}^2} f(u_0) u_0 = \int_{\mathbb{R}^2} g(v_0) u_0.$$

This implies by (3.24) that $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} g(v_n) u_n = \int_{\mathbb{R}^2} g(v_0) u_0$. As a consequence, by (3.26) we obtain

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} |\nabla u_n|^2 + V_0 u_n^2 = \int_{\mathbb{R}^2} f(u_0) u_0 + g(v_0) u_0 - \int_{\mathbb{R}^2} \nabla u_0 \nabla v_0 + V_0 u_0 v_0.$$

Recalling that $\langle \Phi'(u_0, v_0), (u_0, u_0) \rangle = 0$, namely

$$\int_{\mathbb{R}^2} |\nabla u_0|^2 + V_0 u_0^2 = \int_{\mathbb{R}^2} f(u_0) u_0 + g(v_0) u_0 - \int_{\mathbb{R}^2} \nabla u_0 \nabla v_0 + V_0 u_0 v_0,$$

which implies

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} |\nabla u_n|^2 + V_0 u_n^2 = \int_{\mathbb{R}^2} |\nabla u_0|^2 + V_0 u_0^2$$

and hence $u_n \rightarrow u_0$ in $H^1(\mathbb{R}^2)$. Similarly, $v_n \rightarrow v_0$ in $H^1(\mathbb{R}^2)$. \square

Next we prove (i), (iii) of Theorem 2.1 through the following three steps:

- In Proposition 2.7 we prove regularity, namely for any fixed $z = (u, v) \in \mathcal{S}$ we prove that $u, v \in L^\infty(\mathbb{R}^2) \cap C_{loc}^{1,\gamma}(\mathbb{R}^2)$ for some $\gamma \in (0, 1)$;
- In Proposition 2.8 we prove that for any $\{z_n\} \subset \mathcal{S}$, $z_n = (u_n, v_n)$, for which there exists $y_n \in \mathbb{R}^2$ such that $z_n(\cdot + y_n) \rightarrow z_0 \in \mathcal{S}$, one has

$$\sup_{n \geq 1} (\|u_n\|_\infty + \|v_n\|_\infty) < \infty;$$

- Finally, in Proposition 2.9 we prove the following a priori estimates

$$0 < \inf_{z=(u,v) \in \mathcal{S}} \min\{\|u\|_\infty, \|v\|_\infty\} < \sup_{z=(u,v) \in \mathcal{S}} (\|u\|_\infty + \|v\|_\infty) < \infty.$$

Proposition 2.7. *Let $(u, v) \in \mathcal{S}$, then $u, v \in L^\infty(\mathbb{R}^2) \cap C_{loc}^{1,\gamma}(\mathbb{R}^2)$ for some $\gamma \in (0, 1)$.*

Proof. For any $r > 0$, let $B_1 = B_r(0)$, $B_2 = B_{2r}(0)$. Noting that u is a weak solution of the following problem

$$(3.28) \quad -\Delta U + V_0 U = g(v) \text{ in } B_2, \quad U - u \in H_0^1(B_2),$$

by the Pohozaev-Trudinger-Moser inequality one has $g(v) \in L^p(B_2)$ for all $p \geq 2$. By the Calderon-Zygmund inequality, see e.g. [22, Theorem 9.9], one has $u \in W^{2,p}(B_2)$. It follows from classical interior L^p -estimates that

$$(3.29) \quad \|u\|_{W^{2,p}(B_1)} \leq C (\|g(v)\|_{L^p(B_2)} + \|u\|_{L^p(B_2)}),$$

where C only depends on r, p . Meanwhile, by the Sobolev embedding theorem, if $p > 2$ we get that $u \in C^{1,\gamma}(\overline{B_1})$ for some $\gamma \in (0, 1)$ and there exists c (independent of u) such that

$$(3.30) \quad \|u\|_{C^{1,\gamma}(\overline{B_1})} \leq c \|u\|_{W^{2,p}(B_1)}.$$

Next we prove that u vanishes at infinity, namely that for any $\delta > 0$, there exists $R > 0$ such that $|u(x)| \leq \delta$, $\forall |x| \geq R$. Indeed, otherwise there exists $\{x_j\} \subset \mathbb{R}^2$ with $|x_j| \rightarrow \infty$, as $j \rightarrow \infty$ and $\liminf_{j \rightarrow \infty} |u(x_j)| > 0$. Let $u_j(x) = u(x + x_j)$ and $v_j(x) = v(x + x_j)$, then $\|u_j\| = \|u\|$ and

$$(3.31) \quad -\Delta u_j + V_0 u_j = g(v_j), \quad u_j \in H^1(\mathbb{R}^2).$$

We may assume $u_j \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^2)$, we claim that $u_0 \not\equiv 0$. In fact, noting that u_j is a weak solution of (3.28) replacing $g(v)$ by $g(v_j)$, it follows from (3.29) and (3.30) that, up to a subsequence, $u_j \rightarrow u_0$ uniformly in $\overline{\Omega}$. Hence,

$$u_0(0) = \liminf_{j \rightarrow \infty} u_j(0) = \liminf_{j \rightarrow \infty} u(x_j) \neq 0,$$

which implies that $u_0 \not\equiv 0$. On the other hand, for any fixed $R > 0$ and j large enough, we have

$$\begin{aligned} \int_{\mathbb{R}^2} u^2 dx &\geq \int_{B_R(0)} u^2 dx + \int_{B_R(x_j)} u^2 dx \\ &= \int_{B_R(0)} u^2 dx + \int_{B_R(0)} u_j^2 dx \\ &= \int_{B_R(0)} u^2 dx + \int_{B_R(0)} u_0^2 dx + o_j(1), \end{aligned}$$

where $o_j(1) \rightarrow 0$, as $j \rightarrow \infty$. Since R is arbitrary, we get $u_0 \equiv 0$, which is a contradiction. Thus, $u(x) \rightarrow 0$, as $|x| \rightarrow \infty$. Moreover, since $u \in C(B_r)$ for any $r > 0$, we have $u \in L^\infty(\mathbb{R}^2)$. Similarly, $v \in L^\infty(\mathbb{R}^2)$. \square

Proposition 2.8. *Let $z_n = (u_n, v_n) \in \mathcal{S}$ such that $\bar{z}_n = z_n(\cdot + y_n) \rightarrow z_0 = (u_0, v_0) \in \mathcal{S}$ in E , then*

$$\sup_{n \geq 1} (\|u_n\|_\infty + \|v_n\|_\infty) < \infty$$

Proof. Let $\bar{u}_n = u(\cdot + y_n)$, $\bar{v}_n = v(\cdot + y_n)$. Similarly as above, \bar{u}_n is a weak solution of the following problem

$$(3.32) \quad -\Delta U + V_0 U = g(\bar{v}_n) \text{ in } B_2, \quad U - \bar{u}_n \in H_0^1(B_2).$$

Moreover, for any $p \geq 2$ we have

$$(3.33) \quad \|\bar{u}_n\|_{W^{2,p}(B_1)} \leq C (\|g(\bar{v}_n)\|_{L^p(B_2)} + \|\bar{u}_n\|_{L^p(B_2)}),$$

where C only depends on r, p . By the Sobolev embedding theorem, if $p > 2$ we get $\bar{u}_n \in C^{1,\gamma}(\overline{B_1})$ for some $\gamma \in (0, 1)$ and there exists c (independent of n) such that

$$(3.34) \quad \|\bar{u}_n\|_{C^{1,\gamma}(\overline{B_1})} \leq c \|\bar{u}_n\|_{W^{2,p}(B_1)}.$$

Then by (3.33)-(3.34), we get

$$(3.35) \quad \|\bar{u}_n\|_{C^{1,\gamma}(\overline{B_1})} \leq c \left(\|g(\bar{v}_n)\|_{L^p(\mathbb{R}^2)} + \|\bar{u}_n\|_{L^p(\mathbb{R}^2)} \right).$$

By (H1), for $\beta > \alpha_0$ and some $C > 0$, we have $|g(t)| \leq C(|t| + \exp(\beta t^2) - 1), t \in \mathbb{R}$. Recalling that $\bar{v}_n \rightarrow v_0$ in $H^1(\mathbb{R}^2)$, we next prove that

$$(3.36) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} |\exp(p\beta \bar{v}_n^2) - \exp(p\beta v_0^2)| \, dx = 0.$$

In fact, since $v_0 \in L^\infty(\mathbb{R}^2)$, there exists $c > 0$ such that

$$\begin{aligned} & \int_{\mathbb{R}^2} |e^{(p\beta \bar{v}_n^2)} - e^{(p\beta v_0^2)}| \, dx \\ & \leq c \int_{\mathbb{R}^2} e^{(2p\beta |\bar{v}_n - v_0|^2)} |\bar{v}_n^2 - v_0^2| \, dx \\ & = c \int_{\mathbb{R}^2} [e^{(2p\beta |\bar{v}_n - v_0|^2)} - 1] |\bar{v}_n^2 - v_0^2| \, dx + o_n(1) \\ & \leq c \left(\int_{\mathbb{R}^2} [e^{(4p\beta |\bar{v}_n - v_0|^2)} - 1] \, dx \right)^{1/2} \left(\int_{\mathbb{R}^2} |\bar{v}_n^2 - v_0^2|^2 \, dx \right)^{1/2} + o_n(1), \end{aligned}$$

where $o_n(1) \rightarrow 0$, as $n \rightarrow \infty$. From $\|\bar{v}_n - v_0\|_1 \rightarrow 0$, as $n \rightarrow \infty$ and the Pohozaev-Trudinger-Moser inequality, there exists C such that

$$\int_{\mathbb{R}^2} [e^{(4p\beta |\bar{v}_n - v_0|^2)} - 1] \, dx \leq C$$

as n is large enough; thus (3.36) follows.

Recalling that $\bar{z}_n \rightarrow z_0$ in E , by (3.36) $\|g(\bar{v}_n)\|_{L^p(\mathbb{R}^2)} \rightarrow \|g(v_0)\|_{L^p(\mathbb{R}^2)}$ as $n \rightarrow \infty$. Finally we have

$$(3.37) \quad \sup_{n \geq 1} \|\bar{u}_n\|_{C^{1,\gamma}(\overline{B_1})} < \infty.$$

Next we prove that $\bar{u}_n(x) \rightarrow 0$, uniformly as $|x| \rightarrow \infty$. It is enough to prove that for any $\delta > 0$, there exists $R > 0$ such that $|\bar{u}_n(x)| \leq \delta, \forall n \geq 1, |x| \geq R$. Suppose this does not occur, so that there exists $\{x_n\} \subset \mathbb{R}^2$ with $|x_n| \rightarrow \infty$, as $n \rightarrow \infty$ and $\liminf_{n \rightarrow \infty} |\bar{u}_n(x_n)| > 0$. Let $\tilde{u}_n(x) = \bar{u}_n(x + x_n)$ and $\tilde{v}_n(x) = \bar{v}_n(x + x_n)$, then

$$(3.38) \quad -\Delta \tilde{u}_n + V_0 \tilde{u}_n = g(\tilde{v}_n), \quad \tilde{u}_n \in H^1(\mathbb{R}^2).$$

We may assume $\tilde{u}_n \rightharpoonup \tilde{u}_0$ weakly in $H^1(\mathbb{R}^2)$ and we claim $\tilde{u}_0 \not\equiv 0$. For any $n \geq 1$, \tilde{u}_n is a weak solution to the following problem

$$(3.39) \quad -\Delta U + V_0 U = g(\tilde{v}_n) \text{ in } B_2, \quad U - \tilde{u}_n \in H_0^1(B_2).$$

Moreover,

$$(3.40) \quad \|\tilde{u}_n\|_{W^{2,4}(B_1)} \leq C (\|g(\tilde{v}_n)\|_{L^4(B_2)} + \|\tilde{u}_n\|_{L^4(B_2)})$$

where C depends on r only. At the same time, by the Sobolev embedding theorem, we get $\tilde{u}_n \in C^{1,\gamma}(\overline{B_1})$ for some $\gamma \in (0, 1)$ and there exists c (independent of n) such that

$$(3.41) \quad \|\tilde{u}_n\|_{C^{1,\gamma}(\overline{B_1})} \leq c \|\tilde{u}_n\|_{W^{2,4}(B_1)}.$$

Then by (3.40)-(3.41), we get

$$\|\tilde{u}_n\|_{C^{1,\gamma}(\overline{B_1})} \leq c (\|g(v_n)\|_{L^4(\mathbb{R}^2)} + \|u_n\|_{L^4(\mathbb{R}^2)}).$$

Then similar to (3.37), $\sup_{n \geq 1} \|\tilde{u}_n\|_{C^{1,\gamma}(\overline{B_1})} < \infty$. Hence up to a subsequence, $\tilde{u}_n \rightarrow \tilde{u}_0$ uniformly in $\overline{B_1}$. Thus,

$$\tilde{u}_0(0) = \liminf_{n \rightarrow \infty} \tilde{u}_n(0) = \liminf_{n \rightarrow \infty} u_n(x_n) \neq 0,$$

which implies that $\tilde{u}_0 \not\equiv 0$. On the other hand, for any fixed $R > 0$ and j large enough, we have

$$\begin{aligned} o_n(1) + \int_{\mathbb{R}^2} u_0^2 dx &= \int_{\mathbb{R}^2} \bar{u}_n^2 dx \\ &\geq \int_{B_R(0)} \bar{u}_n^2 dx + \int_{B_R(x_n)} \bar{u}_n^2 dx \\ &= \int_{B_R(0)} \bar{u}_n^2 dx + \int_{B_R(0)} \tilde{u}_n^2 dx \\ &= \int_{B_R(0)} u_0^2 dx + \int_{B_R(0)} \tilde{u}_0^2 dx + o_n(1), \end{aligned}$$

where $o_n(1) \rightarrow 0$, as $n \rightarrow \infty$ and we have used the fact that $\bar{u}_n = u_n(\cdot + y_n) \rightarrow u_0$ in $H^1(\mathbb{R}^2)$. Since R is arbitrary, we get $\tilde{u}_0 \equiv 0$, which is a contradiction. Thus, $\bar{u}_n(x) \rightarrow 0$, uniformly as $|x| \rightarrow \infty$, which immediately implies by (3.37) that $\sup_{n \geq 1} \|u_n\|_\infty = \sup_{n \geq 1} \|\bar{u}_n\|_\infty < \infty$. Similarly, $\sup_{n \geq 1} \|v_n\|_\infty < \infty$. \square

Proposition 2.9. *The following a priori estimates hold*

$$(3.42) \quad 0 < \inf_{z=(u,v) \in \mathcal{S}} \min\{\|u\|_\infty, \|v\|_\infty\} < \sup_{z=(u,v) \in \mathcal{S}} (\|u\|_\infty + \|v\|_\infty) < \infty$$

Proof. The upper bound is a consequence of Proposition 2.8 and the fact that \mathcal{S} is compact. In order to prove the lower bound we argue by contradiction and thus assume

$$\inf_{z=(u,v) \in \mathcal{S}} \min\{\|u\|_\infty, \|v\|_\infty\} = 0.$$

Then, there exists $\{z_n\} \subset \mathcal{S}$ such that, without loss of generality, $\|v_n\|_\infty \rightarrow 0$, as $n \rightarrow \infty$. From

$$\int_{\mathbb{R}^2} |\nabla u_n|^2 + V_0 u_n^2 = \int_{\mathbb{R}^2} g(v_n) u_n,$$

by (H1) we have

$$\int_{\mathbb{R}^2} |\nabla u_n|^2 + V_0 u_n^2 \leq o_n(1) \left(\int_{\mathbb{R}^2} v_n^2 \right)^{1/2} \left(\int_{\mathbb{R}^2} u_n^2 \right)^{1/2}$$

and hence $u_n \rightarrow 0$ in $H^1(\mathbb{R}^2)$. From

$$\int_{\mathbb{R}^2} |\nabla v_n|^2 + V_0 v_n^2 = \int_{\mathbb{R}^2} f(u_n) v_n \leq \left(\int_{\mathbb{R}^2} v_n^2 \right)^{1/2} \left(\int_{\mathbb{R}^2} [f(u_n)]^2 \right)^{1/2},$$

together with the fact $u_n \rightarrow 0$ in $H^1(\mathbb{R}^2)$ which implies $\int_{\mathbb{R}^2} [f(u_n)]^2 \rightarrow 0$, we have also $v_n \rightarrow 0$ in $H^1(\mathbb{R}^2)$. Finally, as $(u_n, v_n) \in \mathcal{S}$, we obtain a contradiction from the following

$$0 < c_* = \lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^2} \nabla u_n \nabla v_n + V_0 u_n v_n - \int_{\mathbb{R}^2} F(u_n) + G(v_n) \right) = 0$$

□

In order to complete the proof of Theorem 2.1 it remains to show that ground states vanish at infinity and that enjoy a suitable Pohozaev-type identity in the whole plane; we prove these results in Proposition 2.10 and 2.12 of the next Section.

2.3. Vanishing and Pohozaev-type identity.

Proposition 2.10. (Uniform vanishing) *Let $x_z \in \mathbb{R}^2$ be a maximum point of $|u(x)| + |v(x)|$, $z = (u, v) \in \mathcal{S}$. Then $u(x + x_z) \rightarrow 0$ and $v(x + x_z) \rightarrow 0$, as $|x| \rightarrow \infty$, uniformly for any $(u, v) \in \mathcal{S}$.*

In order to prove Proposition 2.10 we need the following technical lemma

Lemma 2.11. *For any $\{z_n\} \subset \mathcal{S}$, $z_n = (u_n, v_n)$, up to a subsequence, $z_n(\cdot + x_n) \rightarrow z_1$ in E , as $n \rightarrow \infty$, where $\{x_n\} \subset \mathbb{R}^2$ is such that $|u_n(x_n)| + |v_n(x_n)| = \max_{x \in \mathbb{R}^2} (|u_n(x)| + |v_n(x)|)$.*

Proof. We first claim that there exist $\mu > 0$ and $R_1 > 0$ such that

$$(3.43) \quad \lim_{n \rightarrow \infty} \int_{B_{R_1}(x_n)} (u_n^2 + v_n^2) dx \geq \mu.$$

Let us argue by contradiction, indeed if not, for some $\{z_n\} \subset \mathcal{S}$ and any $R > 0$, we get

$$\lim_{n \rightarrow \infty} \int_{B_R(x_n)} (u_n^2 + v_n^2) dx = 0.$$

Let $\hat{u}_n = u_n(\cdot + x_n)$ and $\hat{v}_n = v_n(\cdot + x_n)$, then $\hat{u}_n, \hat{v}_n \rightarrow 0$ in $L_{loc}^2(\mathbb{R}^2)$, as $n \rightarrow \infty$. Similarly as above, \hat{u}_n is a weak solution of the following problem

$$-\Delta U + V_0 U = g(\hat{v}_n) \text{ in } B_2, \quad U - \hat{u}_n \in H_0^1(B_2).$$

By standard elliptic regularity we get $\hat{u}_n \in C^{1,\gamma}(\overline{B_1})$ for some $\gamma \in (0, 1)$ and there exists c (independent of n) such that for $p > 2$,

$$(3.44) \quad \|\hat{u}_n\|_{C^{1,\gamma}(\overline{B_1})} \leq c (\|g(\hat{v}_n)\|_{L^p(\mathbb{R}^2)} + \|\hat{u}_n\|_{L^p(\mathbb{R}^2)}).$$

By Proposition 2.8, $\bar{z}_n \rightarrow z_0$ in E , by (3.36) $\|g(\hat{v}_n)\|_{L^p(\mathbb{R}^2)} = \|g(\bar{v}_n)\|_{L^p(\mathbb{R}^2)} \rightarrow \|g(v_0)\|_{L^p(\mathbb{R}^2)}$, as $n \rightarrow \infty$. Then we have

$$(3.45) \quad \sup_{n \geq 1} \|\hat{u}_n\|_{C^{1,\gamma}(\overline{B_1})} < \infty,$$

which implies by $\hat{u}_n \rightarrow 0$ in $L^2(B_1)$ that $\hat{u}_n \rightarrow 0$ uniformly in B_1 . In particular, $\hat{u}_n(0) = u_n(x_n) \rightarrow 0$. Similarly, we have $\hat{v}_n(0) = v_n(x_n) \rightarrow 0$. Finally we obtain

$$\lim_{n \rightarrow \infty} \max_{x \in \mathbb{R}^2} (|u_n(x)| + |v_n(x)|) = \lim_{n \rightarrow \infty} (|u_n(x_n)| + |v_n(x_n)|) = 0,$$

which implies

$$\lim_{n \rightarrow \infty} \min\{\|u_n\|_\infty, \|v_n\|_\infty\} = 0$$

and thus a contradiction.

Now by (3.43) $\lim_{n \rightarrow \infty} \int_{B_{R_1}(0)} (\hat{u}_n^2 + \hat{v}_n^2) dx \geq \mu$ which combined with the local compactness of the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$, yields up to a subsequence, $z_n(\cdot + x_n) = (\hat{u}_n + \hat{v}_n) \rightharpoonup z_1 \neq 0$ in E and $z_n(\cdot + x_n) \rightarrow z_1$ a.e. in \mathbb{R}^2 , as $n \rightarrow \infty$. Then arguing as in Proposition 2.5-2.6, we get $z_1 \in \mathcal{S}$ and $z_n(\cdot + x_n) \rightarrow z_1$ in E , as $n \rightarrow \infty$, and this completes the proof. \square

Proof of Proposition 2.10.

Next let us prove that for any $\delta > 0$, there exists $R > 0$ such that $|u(x + x_z)| + |v(x + x_z)| \leq \delta$, $|x| \geq R$ for any $z = (u, v) \in \mathcal{S}$, where $x_z \in \mathbb{R}^2$ is a maximum point of $|u(x)| + |v(x)|$. If not, there exist $z_n = (u_n, v_n) \in \mathcal{S}$ and $\{x_n\} \subset \mathbb{R}^2$ such that $|x_n| \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\liminf_{n \rightarrow \infty} (|u_n(x_n + x_{z_n})| + |v_n(x_n + x_{z_n})|) > 0,$$

where $x_{z_n} \in \mathbb{R}^2$ is a maximum point of $|u_n(x)| + |v_n(x)|$. Without loss of generality, we may assume $\liminf_{n \rightarrow \infty} |u_n(x_n + x_{z_n})| > 0$. Let $\tilde{u}_n(x) = u_n(x + x_n + x_{z_n})$ and $\tilde{v}_n(x) = v_n(x + x_n + x_{z_n})$. Assume $\tilde{u}_n \rightharpoonup \tilde{u}_0$ weakly in $H^1(\mathbb{R}^2)$, in the following we claim $\tilde{u}_0 \neq 0$. Indeed, by Lemma 2.11, up to a subsequence, there exists $z \in \mathcal{S}$ such that $(u_n(\cdot + x_{z_n}), v_n(\cdot + x_{z_n})) \rightarrow z$ strongly in E . Then as in the proof of the above Lemma, by the elliptic estimates, up to a subsequence, for some $\tilde{u}_0 \in H^1(\mathbb{R}^2)$ and $\gamma \in (0, 1)$, $\tilde{u}_n \rightarrow \tilde{u}_0$ in $C_{loc}^{1,\gamma}(\mathbb{R}^2)$, as $n \rightarrow \infty$. Hence,

$$\tilde{u}_0(0) = \liminf_{n \rightarrow \infty} \tilde{u}_n(0) = \liminf_{n \rightarrow \infty} u_n(x_n + x_{z_n}) \neq 0,$$

which implies that $\tilde{u}_0 \neq 0$. On the other hand, proceeding as in Proposition 2.8, we get $\tilde{u}_0 \equiv 0$, which is a contradiction. \square

Proposition 2.12. (Pohozaev-type identity) *For any $z = (u, v) \in \mathcal{S}$, the following Pohozaev-type identity holds true*

$$(3.46) \quad \int_{\mathbb{R}^2} (F(u) + G(v) - V_0 uv) dx = 0.$$

Proof. By the proof of Proposition 2.7 we know $u, v \in W_{\text{loc}}^{2,p}(\mathbb{R}^2)$ for any $p \geq 2$. Then $\Delta u = V_0 u - g(v)$ a.e. in \mathbb{R}^2 and $\Delta v = V_0 v - f(u)$ a.e. in \mathbb{R}^2 . Following [26, 35] we get

$$(3.47) \quad \oint_{\partial B_r} \nabla u \nabla v \cdot (x, \mathbf{n}) \, ds - \oint_{\partial B_r} \left(\sum_{i,j=1}^2 x_j \left(\frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \frac{\partial v}{\partial x_j} \frac{\partial u}{\partial x_i} \right), \mathbf{n} \right) \, ds \\ = 2 \int_{B_r} (V_0 uv - F(u) - G(v)) \, dx,$$

where $B_r(0) := \{x \in \mathbb{R}^2 : |x| < r\}$, $r > 0$ and \mathbf{n} is the outward normal of ∂B_r at x . From $\nabla u, \nabla v \in L^2(\mathbb{R}^2)$, by virtue of the coarea formula, there exists r_n such that $r_n \rightarrow \infty$ and

$$r_n \oint_{\partial B_{r_n}} \left| \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \right| \, ds \rightarrow 0, \text{ for any } i, j = 1, 2.$$

As a consequence as $n \rightarrow \infty$,

$$\left| \oint_{\partial B_{r_n}} \nabla u \nabla v \cdot (x, \mathbf{n}) \, ds \right| \leq r_n \oint_{\partial B_{r_n}} |\nabla u \nabla v| \, ds \rightarrow 0$$

and hence

$$\oint_{\partial B_r} \left(\sum_{i,j=1}^2 x_j \left(\frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \frac{\partial v}{\partial x_j} \frac{\partial u}{\partial x_i} \right), \mathbf{n} \right) \, ds \rightarrow 0.$$

Then, let $r = r_n$ in (3.47) to get, as $n \rightarrow \infty$, identity (3.46). \square

2.4. Sign and symmetry properties. This section is devoted to proving Theorem 2.2. To investigate positivity and radial symmetry of ground state solutions to (2.1), without loss generality, throughout this section we assume that f, g are odd symmetric functions. Let

$$\kappa := \sup\{\|u\|_\infty, \|v\|_\infty : (u, v) \in S\} < \infty$$

by Theorem 2.1. By (H1) and (H6), there exist small $a_0, b_0 \in [0, 1)$ and $k_1, k_2 > 0$ with

$$k_1 = \max_{a_0 < |t| \leq \kappa} |f(t)|/|t|^q, \quad k_2 = \max_{b_0 < |t| \leq \kappa} |g(t)|/|t|^p,$$

such that $f(t) \leq t$, for $t \in [0, a_0]$ and $g(t) \leq t$, for $t \in [0, b_0]$. Moreover, $f(a_0) = k_1 a_0^q$ and $g(b_0) = k_2 b_0^p$. In fact, if $\limsup_{t \rightarrow 0} |f(t)|/|t|^q < \infty$, we can choose $a_0 = 0$, otherwise there exists $a_0 \in (0, 1)$ such that $f(a_0)/a_0^q = \max_{t \in [a_0, \kappa]} f(t)/t^q$. Let

$$f_k(t) = \begin{cases} f(t), & \text{if } t \in [0, a_0] \\ \min\{f(t), k_1 t^q\}, & \text{if } t \in (a_0, \infty) \end{cases}$$

and $f_k(t) = -f_k(-t)$ for $t \leq 0$ and similarly for g . Then, $f_k, g_k \in C(\mathbb{R}, \mathbb{R})$ and $f_k(t) = f(t), g_k(t) = g(t)$ if $|t| \leq \kappa$, $0 < f_k(t) \leq f(t), 0 < g_k(t) \leq g(t)$ for all $t > 0$. At the same time, there exists $\beta > 0$ such that

$$(3.48) \quad \begin{cases} |f_k(t)| \geq \beta |t|^q \text{ and } |g_k(t)| \geq \beta |t|^p, & \text{for any } t \in \mathbb{R} \\ |f_k(t)| = |f(t)| \leq |t| \text{ if } |t| \leq a_0, & |g_k(t)| = |g(t)| \leq |t| \text{ if } |t| \leq b_0 \\ |f_k(t)| \leq k_1 |t|^q \text{ if } |t| \geq a_0, & |g_k(t)| \leq k_2 |t|^p \text{ if } |t| \geq b_0. \end{cases}$$

Moreover, it is easy to check that f_k, g_k satisfy (H1), (H4) and

$$(3.49) \quad 0 < 2F_k(t) \leq f_k(t)t, \quad 0 < 2G_k(t) \leq g_k(t)t, \quad t \neq 0,$$

$$(3.50) \quad \lim_{|t| \rightarrow \infty} \frac{F_k(t)}{t^2} = \infty, \quad \lim_{|t| \rightarrow \infty} \frac{G_k(t)}{t^2} = \infty,$$

where $F_k(t) = \int_0^t f_k(\tau) d\tau$ and $G_k(t) = \int_0^t g_k(\tau) d\tau$.

Now consider the truncated problem

$$(3.51) \quad \begin{cases} -\Delta u + V_0 u = g_k(v) \\ -\Delta v + V_0 v = f_k(u) \end{cases}$$

whose associated energy functional is

$$\Phi_k(z) := \int_{\mathbb{R}^2} (\nabla u \nabla v + V_0 uv) dx - \int_{\mathbb{R}^2} (F_k(u) + G_k(v)) dx, \quad z = (u, v) \in E.$$

Recall the generalized Nehari Manifold

$$\mathcal{N}_k := \{z \in E \setminus E^- : \langle \Phi'_k(z), z \rangle = 0, \langle \Phi'_k(z), \varphi \rangle = 0 \text{ for all } \varphi \in E^-\}$$

and the least energy

$$c_*^k := \inf_{z \in \mathcal{N}_k} \Phi_k(z).$$

Noting that for any $(u, v) \in \mathcal{S}$, (u, v) is a solution to (3.51), hence $c_*^k \leq c_*$. For $z \in E \setminus E^-$, set

$$\hat{E}(z) = E^- \oplus \mathbb{R}^+ z = E^- \oplus \mathbb{R}^+ z^+.$$

From [16, 33, 34] we have

Lemma 2.13.

- 1) For any $z \in \mathcal{N}_k$, $\Phi_k|_{\hat{E}(z)}$ has a unique maximum point which occurs exactly at z ;
- 2) For any $z \in E \setminus E^-$, the set $\hat{E}(z)$ intersects \mathcal{N}_k at exactly one point $\hat{m}_k(z)$, which is the unique global maximum point of $\Phi_k|_{\hat{E}(z)}$;
- 3)

$$c_*^k := \inf_{z \in E \setminus E^-} \max_{\omega \in \hat{E}(z)} \Phi_k(\omega).$$

From $0 \leq G_k(t) \leq G(t)$ and $0 \leq F_k(t) \leq F(t)$ for any $t \in \mathbb{R}$, we have

$$c_*^k \geq \inf_{z \in E \setminus E^-} \max_{\omega \in \hat{E}(z)} \Phi(\omega) = c_*.$$

thus $c_*^k = c_* > 0$.

Next define

$$\hat{m}_k : z \in E \setminus E^- \mapsto \hat{m}_k(z) \in \hat{E}(z) \cap \mathcal{N}_k.$$

There exists $\delta > 0$ such that $\|z^+\|_\varepsilon \geq \delta$ for all $z \in \mathcal{N}_k$; in particular one has

$$\|\hat{m}_k(z)^+\|_\varepsilon \geq \delta \quad \text{for all } z \in E \setminus E^-.$$

Moreover, for each compact subset $\mathcal{W} \subset E \setminus E^-$, there exists a constant $C_{\mathcal{W}} > 0$ such that

$$\|\hat{m}(z)\| \leq C_{\mathcal{W}} \quad \text{for all } z \in \mathcal{W}.$$

Define

$$S^+ := \{z \in E^+ : \|z\| = 1\},$$

then, S^+ is a C^1 -submanifold of E^+ and the tangent manifold of S^+ at $z \in S^+$ is given by

$$T(S^+) = \{\omega \in E^+ : (\omega, z) = 0\}.$$

Let

$$m_k := \hat{m}_k|_{S^+} : S^+ \longrightarrow \mathcal{N}_k,$$

then \hat{m}_k is continuous and m_k is a homeomorphism between S^+ and \mathcal{N}_k . Define

$$\Psi_k : S^+ \longrightarrow \mathbb{R}, \Psi_k(z) := \Phi_k(m_k(z)), z \in S^+$$

then, by [34, Corollary 4.3] we have

Proposition 2.14.

1) $\Psi_k \in C^1(S^+, \mathbb{R})$ and

$$\langle \Psi'_k(z), \omega \rangle = \|m_k(z)^+\| \langle \Phi'_k(m_k(z)), \omega \rangle, \text{ for all } \omega \in T_z(S^+);$$

2) If $\{\omega_n\} \subset S^+$ is a Palais-Smale sequence for Ψ_k , then $\{m_k(\omega_n)\} \subset \mathcal{N}_k$ is a Palais-Smale sequence for Φ_k . Namely, if $\Psi_k(\omega_n) \rightarrow d$ for some $d > 0$ and $\|\Psi'_k(\omega_n)\|_* \rightarrow 0$ as $n \rightarrow \infty$, then $\Phi_k(m_k(\omega_n)) \rightarrow d$ and $\|\Phi'_k(m_k(\omega_n))\| \rightarrow 0$ as $n \rightarrow \infty$, where

$$\|\Psi'_k(\omega_n)\|_* = \sup_{\substack{\phi \in T_{\omega_n}(S^+) \\ \|\phi\|=1}} \langle \Psi'_k(\omega_n), \phi \rangle \text{ and } \|\Phi'_k(m_k(\omega_n))\| = \sup_{\substack{\phi \in E \\ \|\phi\|=1}} \langle \Phi'_k(m_k(\omega_n)), \phi \rangle;$$

3) $\omega \in S^+$ is a critical point of Ψ_k if and only if $m_k(\omega) \in \mathcal{N}_k$ is a critical point of Φ_k ;

4) $\inf_{S^+} \Psi_k = \inf_{\mathcal{N}_k} \Phi_k$.

It follows from the Ekeland Variational Principle (see [20, Theorem 3.1]) that there exists $\{z_n^k\} \subset \mathcal{N}_k$ such that

$$(3.52) \quad \Phi_k(z_n^k) \rightarrow c_* > 0 \text{ and } \Phi'_k(z_n^k) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Next we prove that $\{z_n^k\}$ is uniformly bounded in E . Precisely we have the following

Lemma 2.15. *There exists $C > 0$ such that $\|z_n^k\| = \|(u_n^k, v_n^k)\| \leq C$, for all $n \in \mathbb{N}$.*

Proof. Let $z_n^k = z_n^+ + z_n^-$, where $z_n^+ \in E^+$, $z_n^- \in E^-$. Noting that $z_n^k \in \mathcal{N}_k$, we have $\|z_n^+\|^2 \geq \|z_n^k\|^2/2$ for all $n \in \mathbb{N}$. Let $w_n^k = w_n^+ + w_n^- = z_n^k/\|z_n^k\|$, where $w_n^+ \in \hat{E}(z_n^k) \subset E^+$, $w_n^- \in E^-$ and $w_n^+ = (\tilde{w}_n, \tilde{w}_n)$, then $\|w_n^+\|^2 \geq 1/2$. By Lemma 2.13, for some $R > 2\sqrt{c_*}$, we have

$$\begin{aligned} c_* + o_n(1) &= \Phi_k(z_n^k) = \max_{w \in \hat{E}(z_n^k)} \Phi_k(w) \geq \Phi_k(Rw_n^+) \\ &\geq R^2/4 - \int_{\mathbb{R}^2} F_k(R\tilde{w}_n) + G_k(R\tilde{w}_n), \end{aligned}$$

which implies

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} F_k(R\tilde{w}_n) + G_k(R\tilde{w}_n) > 0.$$

By Lions' Lemma, up to translations, $\tilde{w}_n \rightarrow w \neq 0$ weakly in $H^1(\mathbb{R}^2)$ as $n \rightarrow \infty$. Assume that $w_n^k \rightarrow (u, v)$ weakly in $H^1(\mathbb{R}^2)$ as $n \rightarrow \infty$, then $u + v = 2w$. If $\|z_n^k\| \rightarrow \infty$ as $n \rightarrow \infty$, then $u_n^k(x) \rightarrow \infty$ if $u(x) \neq 0$ as $n \rightarrow \infty$ and by Fatou's Lemma and (3.50),

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} \left(\frac{F_k(u_n^k)}{\|z_n^k\|^2} + \frac{G_k(v_n^k)}{\|z_n^k\|^2} \right) = +\infty,$$

which yields $\Phi_k(z_n^k) \rightarrow -\infty$ as $n \rightarrow \infty$. This is a contradiction and therefore $\{z_n^k\}$ stays bounded in E . \square

Up to a subsequence, we may assume $z_n^k \rightharpoonup z^k$ weakly in E , as $n \rightarrow \infty$. It is standard to check that $\Phi'_k(z^k) = 0$.

Proposition 2.16. *The truncated problem (3.51) admits a ground state solution.*

Proof. If $z^k \neq 0$, then by (3.50) and Fatou's Lemma one has

$$\begin{aligned} c_* + o_n(1) &= \Phi_k(z_n^k) - \frac{1}{2} \langle \Phi'_k(z_n^k), z_n^k \rangle \\ &= \int_{\mathbb{R}^2} \frac{1}{2} f_k(u_n^k) u_n^k - F_k(u_n^k) + \int_{\mathbb{R}^2} \frac{1}{2} g_k(v_n^k) v_n^k - G_k(v_n^k) \\ &\geq \int_{\mathbb{R}^2} \frac{1}{2} f_k(u^k) u^k - F_k(u^k) + \int_{\mathbb{R}^2} \frac{1}{2} g_k(v^k) v^k - G_k(v^k) + o_n(1) \\ &= \Phi_k(z^k) - \frac{1}{2} \langle \Phi'_k(z^k), z^k \rangle + o_n(1) \\ &= \Phi_k(z^k) \geq c_* + o_n(1). \end{aligned}$$

from which z^k is a ground state solution to (3.51).

If $z^k = 0$, we claim there exist $\nu > 0$, $R_0 > 0$ and $\{y_n\} \subset \mathbb{R}^2$ such that

$$(3.53) \quad \lim_{n \rightarrow \infty} \int_{B_{R_0}(y_n)} (|u_n^k|^2 + |v_n^k|^2) dx \geq \nu.$$

Suppose the claim holds true and set $\tilde{u}_n^k(\cdot) := u_n^k(\cdot + y_n)$ and $\tilde{v}_n^k(\cdot) := v_n^k(\cdot + y_n)$, so that

$$(3.54) \quad \lim_{n \rightarrow \infty} \int_{B_{R_0}(0)} (|\tilde{u}_n^k|^2 + |\tilde{v}_n^k|^2) dx \geq \nu,$$

and $\Phi_k(\tilde{z}_n^k) \rightarrow c_* > 0$ and $\Phi'_k(\tilde{z}_n^k) \rightarrow 0$, as $n \rightarrow \infty$ where $\tilde{z}_n^k = (\tilde{u}_n^k, \tilde{v}_n^k)$. Clearly $\{\tilde{z}_n^k\}$ is bounded in E and up to a subsequence, by (3.54) we may assume that $\tilde{z}_n^k \rightarrow \tilde{z}^k \neq 0$ weakly in E to a ground state solution of (3.51).

Hence let us prove by contradiction the claim (3.53). Indeed, if (3.53) does not hold we have

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^2} \int_{B_R(y)} (|u_n^k|^2 + |v_n^k|^2) dx = 0 \quad \text{for all } R > 0,$$

then by Lions's Lemma, $u_n^k \rightarrow 0$, $v_n^k \rightarrow 0$ strongly in $L^s(\mathbb{R}^2)$ for any $s > 2$. By (H1) and (3.48) we have

$$\int_{\mathbb{R}^2} (|\nabla u_n^k| + V_0 |u_n^k|) dx = \int_{\mathbb{R}^2} g_k(v_n^k) u_n^k dx \rightarrow 0, \quad n \rightarrow \infty.$$

Namely, $u_n^k \rightarrow 0$ strongly in E , as $n \rightarrow \infty$. It follows that

$$\int_{\mathbb{R}^2} (|\nabla v_n^k| + V_0 |v_n^k|) dx = \int_{\mathbb{R}^2} f_k(u_n^k) v_n^k dx \rightarrow 0, \quad n \rightarrow \infty.$$

Namely, $v_n^k \rightarrow 0$ strongly in E , as $n \rightarrow \infty$. So we get $c_* + o_n(1) = \Phi_k(z_n^k) \rightarrow 0$, as $n \rightarrow \infty$, which is a contradiction. \square

Denote by \mathcal{S}_k the set of ground state solutions to system (3.51), then $\mathcal{S}_k \neq \emptyset$. Similarly as above, for any $z = (u, v) \in \mathcal{S}_k$, $u, v \in L^\infty(\mathbb{R}^2) \cap C_{loc}^{1,\gamma}(\mathbb{R}^2)$ for some $\gamma \in (0, 1)$. Recalling that $c_* = c_*^k$, we get $\mathcal{S} \subseteq \mathcal{S}_k$. In order to prove the reverse inclusion let us recall the following results from [16]

Lemma 2.17. [16] *With the assumptions in Theorem 2.1, we have:*

- 1) *for any $z \in \mathcal{N}$, $\Phi|_{\hat{E}(z)}$ admits a unique maximum point which is precisely at z ;*
- 2) *for any $z \in E \setminus E^-$, the set $\hat{E}(z)$ intersects \mathcal{N} at exactly one point $\hat{m}(z)$, which is the unique globally maximum point of $\Phi|_{\hat{E}(z)}$;*
- 3)

$$c_* = \inf_{z \in E \setminus E^-} \max_{\omega \in \hat{E}(z)} \Phi(\omega).$$

Let $m := \hat{m}|_{S^+} : S^+ \mapsto \mathcal{N}$ and

$$\Psi : S^+ \mapsto \mathbb{R}, \quad \Psi(z) := \Phi(m(z)), \quad z \in S^+,$$

then \hat{m} is continuous and m is a homeomorphism between S^+ and \mathcal{N} . As in [34], m is invertible and the inverse is given by

$$m^{-1}(z) = \frac{z^+}{\|z\|}, \quad z = z^+ + z^- \in \mathcal{N}, \quad z^+ \in E^+, \quad z^- \in E^-.$$

Similar to Proposition 2.14, we have

Proposition 2.18.

- 1) $\Psi \in C^1(S^+, \mathbb{R})$ and

$$\langle \Psi'(z), \omega \rangle = \|m(z)^+\| \langle \Phi'(m(z)), \omega \rangle \quad \text{for all } \omega \in T_z(S^+);$$

- 2) *If $\{\omega_n\} \subset S^+$ is a Palais-Smale sequence for Ψ , then $\{m(\omega_n)\} \subset \mathcal{N}$ is a Palais-Smale sequence for Φ . Namely, if $\Psi(\omega_n) \rightarrow d$ for some $d > 0$ and $\|\Psi'(\omega_n)\|_* \rightarrow 0$ as $n \rightarrow \infty$, then $\Phi(m(\omega_n)) \rightarrow d$ and $\|\Phi'(m(\omega_n))\| \rightarrow 0$ as $n \rightarrow \infty$, where*

$$\|\Psi'(\omega_n)\|_* = \sup_{\substack{\phi \in T_{\omega_n}(S^+) \\ \|\phi\|=1}} \langle \Psi'(\omega_n), \phi \rangle \quad \text{and} \quad \|\Phi'(m(\omega_n))\| = \sup_{\substack{\phi \in E \\ \|\phi\|=1}} \langle \Phi'(m(\omega_n)), \phi \rangle;$$

- 3) $\omega \in S^+$ is a critical point of Ψ if and only if $m(\omega) \in \mathcal{N}$ is a critical point of Φ ;
- 4) $\inf_{S^+} \Psi = \inf_{\mathcal{N}} \Phi$.

Proposition 2.19.

$$\mathcal{S}_k = \mathcal{S}.$$

Proof. For any $z^k \in \mathcal{S}_k$, we know $z^k \in \mathcal{N}_k$, by Lemma 2.13 $\Phi_k|_{\hat{E}(z)}$ admits a unique maximum point at z^k and

$$c_*^k := \inf_{z \in E \setminus E^-} \max_{\omega \in \hat{E}(z)} \Phi_k(\omega) = \max_{\omega \in \hat{E}(z^k)} \Phi_k(\omega).$$

Since $z^k \in E \setminus E^-$, by Lemma 2.17 the set $\hat{E}(z^k)$ intersects \mathcal{N} just at one point $\hat{m}(z^k)$, which is the unique global maximum of $\Phi|_{\hat{E}(z^k)}$. Let $\hat{m}(z^k) = (\hat{u}^k, \hat{v}^k)$, then by $0 \leq f_k(t) \leq f(t)$ and $0 \leq g_k(t) \leq g(t)$, for $t \geq 0$ we have

$$\begin{aligned} c_*^k &= \max_{\omega \in \hat{E}(z^k)} \Phi_k(\omega) \geq \Phi_k(\hat{m}(z^k)) \\ &= \Phi(\hat{m}(z^k)) + \int_{\mathbb{R}^2} [F(\hat{u}_k) - F_k(\hat{u}_k)] dx + \int_{\mathbb{R}^2} [G(\hat{v}_k) - G_k(\hat{v}_k)] dx \\ &= \max_{\omega \in \hat{E}(z^k)} \Phi(\omega) + \int_{\mathbb{R}^2} [F(\hat{u}_k) - F_k(\hat{u}_k)] dx + \int_{\mathbb{R}^2} [G(\hat{v}_k) - G_k(\hat{v}_k)] dx \\ &\geq \inf_{z \in E \setminus E^-} \max_{\omega \in \hat{E}(z)} \Phi(\omega) \geq c_*, \end{aligned}$$

which implies $F(\hat{u}_k(x)) \equiv F_k(\hat{u}_k(x))$ and $G(\hat{v}_k(x)) \equiv G_k(\hat{v}_k(x))$ for all $x \in \mathbb{R}^2$ and

$$\max_{\omega \in \hat{E}(z^k)} \Phi_k(\omega) = \Phi_k(\hat{m}(z^k)) = \Phi(\hat{m}(z^k)) = c_*.$$

Then $\Psi(m^{-1}(\hat{m}(z^k))) := \Phi(\hat{m}(z^k)) = c_*$. Notice that $m^{-1}(\hat{m}(z^k)) \in S^+$. Then, by Proposition 2.18, $m^{-1}(\hat{m}(z^k))$ is a minimizer of Ψ on the C^1 -manifold S^+ . Thus

$$\langle \Psi'(m^{-1}(\hat{m}(z^k))), \omega \rangle = 0 \quad \text{for all } \omega \in T_{m^{-1}(\hat{m}(z^k))}(S^+).$$

It follows from 3) of Proposition 2.18 that $\Phi'(\hat{m}(z^k)) = 0$, which yields $\hat{m}(z^k) \in \mathcal{S}$. By uniqueness of the global maximum point of $\Phi_k|_{\hat{E}(z^k)}$, we get $z^k = \hat{m}(z^k)$ and hence $z^k \in \mathcal{S}$. Therefore, $\mathcal{S}_k = \mathcal{S}$. \square

In the last part of this section, in the spirit of [4] we prove that $uv > 0$ in \mathbb{R}^2 for any $z = (u, v) \in \mathcal{S}_k$.

Let $h(s) := g_k^{-1}(s)$ and H denote the primitive function of h . By (3.48), for some $c, C > 0$,

$$(3.55) \quad \begin{cases} h(s)s \leq C|s|^{(p+1)/p} & \text{for } s \in \mathbb{R}, \\ h(s)s \geq s^2/2 & \text{if } |s| \leq g(b_0), \\ h(s)s \geq c|s|^{(p+1)/p} & \text{if } |s| > g(b_0). \end{cases}$$

and clearly the same estimates hold for $H(s)$ as well. Consider the Schrödinger operator $L := -\Delta + V_0$ and the Sobolev space $W^{2,(p+1)/p}(\mathbb{R}^2)$ endowed with the norm

$$|||u||| = \left(\int_{\mathbb{R}^2} |Lu|^{\frac{p+1}{p}} dx \right)^{\frac{p}{p+1}}.$$

The following embeddings hold

$$W^{2, \frac{s+1}{s}}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N), \text{ for any } r \geq \frac{s+1}{s}, s > 1, \text{ if } s(N-2) \leq 2,$$

in particular $W^{2,(p+1)/p}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2) \cap L^{p+1}(\mathbb{R}^2) \cap L^{q+1}(\mathbb{R}^2)$. For $u \in W^{2,(p+1)/p}(\mathbb{R}^2)$, define

$$J_k(u) = \int_{\mathbb{R}^2} H(Lu) - F_k(u) dx$$

then J_k is of class C^1 and

$$\langle J'_k(u), \varphi \rangle = \int_{\mathbb{R}^2} (h(Lu)L(\varphi) - f(u)\varphi) dx, \quad u, \varphi \in W^{2,(p+1)/p}(\mathbb{R}^2).$$

Proposition 2.20. *$(u, v) \in E$ is a critical point of Φ_k if and only if u is a critical point of J_k and $v = h(Lu)$. Moreover, one has $\Phi_k(u, v) = J_k(u)$.*

Define

$$c_1(\mathbb{R}^2) = \inf_{u \in \mathcal{N}_J} J_k(u), \quad \text{where } \mathcal{N}_J := \{u \in W^{2,(p+1)/p}(\mathbb{R}^2) \setminus \{0\} : \langle J'_k(u), u \rangle = 0\},$$

which under our assumptions might not be well defined. We overcome this difficulty by considering an approximation via bounded domains. Precisely, for any $R > 0$ let us consider the problem

$$(3.56) \quad \begin{cases} -\Delta u + V_0 u = g_k(v) \\ -\Delta v + V_0 v = f_k(u) \end{cases}$$

$u, v \in H_0^1(B_R(0))$ whose associated energy functional is

$$I_R(z) := \int_{B_R(0)} (\nabla u \nabla v + V_0 uv) dx - (F_k(u) + G_k(v)) dx,$$

where $z = (u, v) \in E_R := H_0^1(B_R(0)) \times H_0^1(B_R(0))$.

We can define as above $E_R^+, E_R^-, \hat{E}_R(z)$ and

$$\mathcal{N}_R := \{z \in E_R \setminus E_R^- : \langle I'_R(z), z \rangle = 0, \langle I'_R(z), \phi \rangle = 0 \text{ for all } \phi \in E_R^-\}.$$

Denote by $c_*(B_R(0))$ the corresponding least energy associated to the energy functional I_R . Similar to Lemma 2.15, every Palais-Smale sequence for I_R is bounded in E_R . Then $c_*(B_R(0))$ is the ground state critical level associated to I_R . Moreover,

$$c_*(B_R(0)) = \inf_{z \in E_R \setminus E_R^-} \max_{\omega \in \hat{E}_R(z)} I_R(\omega).$$

Remark 2.21. *If $z = (u, v) \in \mathcal{N}_R$, we have $\langle I'_R(z), (\varphi, -\varphi) \rangle = 0$ for all $\varphi \in H_0^1(B_R(0))$. In general, $\langle I'_R(z), (\varphi, -\varphi) \rangle = 0$ does not hold for all $\varphi \in H^1(\mathbb{R}^2)$. Then, \mathcal{N}_R is not a subset of \mathcal{N} , so it is not clear if $c_*(B_R(0))$ is greater than c_* .*

Let

$$X_R = W^{2,(p+1)/p}(B_R(0)) \cap W_0^{1,(p+1)/p}(B_R(0))$$

endowed with the norm

$$|||u||| = \left(\int_{B_R} |Lu|^{\frac{p+1}{p}} dx \right)^{\frac{p}{p+1}}$$

and

$$J_R(u) = \int_{B_R(0)} H(Lu) - F_k(u) dx, \quad u \in X_R.$$

Proposition 2.22. *$z = (u, v) \in E_R$ is a critical point of I_R if and only if u is a critical point of J_R and $v = h(Lu)$. Moreover, $I_R(u, v) = J_R(u)$.*

Let

$$\mathcal{N}_{J_R} := \{u \in X_R \setminus \{0\} : \langle J'_R(u), u \rangle = 0\}, \quad c_1(B_R(0)) := \inf_{u \in \mathcal{N}_{J_R}} J_R(u).$$

Notice that \mathcal{N}_{J_R} might not be a C^1 -manifold, so that we next borrow some ideas of [34] to overcome this difficulty and prove the existence of ground states corresponding to the functional J_R on \mathcal{N}_{J_R} for any R . Then by passing to the limit, we show that $c_1(\mathbb{R}^2)$ is the ground state critical value.

Lemma 2.23. *For any $u \in X_R \setminus \{0\}$, $J_R(tu) \rightarrow -\infty$, as $t \rightarrow +\infty$ and the set \mathbb{R}^+u intersects \mathcal{N}_{J_R} at exactly one point denoted by $\hat{m}_R(u)$, which is the unique global maximum point of $J_R(tu)$, for $t > 0$. In particular, $\hat{m}_R(u) = 1$ if and only if $u \in \mathcal{N}_{J_R}$. Moreover, there exist $a_R, b_R > 0$ such that*

$$|||u||| \geq a_R \text{ for any } u \in \mathcal{N}_{J_R} \text{ and } c_1(B_R(0)) \geq b_R.$$

Proof. Step 1. By (3.48) and (3.55), for any $u \in X_R \setminus \{0\}$ and $t > 0$,

$$J_R(tu) \leq Ct^{(p+1)/p} \int_{B_R(0)} |Lu|^{(p+1)/p} - \frac{q+1}{q} \beta t^{q+1} \int_{B_R(0)} |u|^{q+1} \rightarrow -\infty, \quad t \rightarrow +\infty,$$

and for any $\gamma > 0$ small, there exists $c_\gamma > 0$ such that

$$\begin{aligned} J_R(tu) &\geq \frac{t^2}{2} \int_{\{|Lu| \leq g(b_0)\}} |Lu|^2 + ct^{(p+1)/p} \int_{\{|Lu| > g(b_0)\}} |Lu|^{(p+1)/p} \\ &\quad - \gamma t^2 \int_{B_R(0)} |u|^2 - c_\gamma t^{q+1} \int_{B_R(0)} |u|^{q+1} > 0, \quad |t| \ll 1, \end{aligned}$$

where

$$\{|Lu| \leq g(b_0)\} := \{x \in B_R(0) : |Lu(x)| \leq g(b_0)\}.$$

For any $u \in \mathcal{N}_{J_R}$, let $\theta(t) = J_R(tu)$, then $\theta(0) = 0$ and $\theta'(1) = 0$. Recalling that $g_k(s)/s$ is strictly increasing for $s > 0$, $h(s)/s$ is strictly decreasing for $s > 0$. Obviously, $Lu = 0$ if and only if $u = 0$. Then for any $t > 1$, thanks to (H4), (H6),

$$\begin{aligned} \theta'(t) &= \int_{B_R(0)} h(tLu)Lu - \int_{B_R(0)} f_k(tu)u \\ &= \int_{B_R(0)} h(t|Lu|)|Lu| - \int_{B_R(0)} f_k(t|u|)|u| \\ &= \int_{B_R(0)} \frac{h(t|Lu|)}{t|Lu|} t|Lu|^2 - \int_{B_R(0)} \frac{f_k(t|u|)}{t|u|} t|u|^2 \\ &< t \int_{B_R(0)} h(|Lu|)|Lu| - t \int_{B_R(0)} f_k(|u|)|u| \\ &= t \int_{B_R(0)} h(Lu)Lu - t \int_{B_R(0)} f_k(u)u = 0. \end{aligned}$$

Similarly, $\theta'(t) > 0$ for $t < 1$. Namely, $J_R(u) = \max_{t \geq 0} J_R(tu)$. Similarly, for any $u \in X_R \setminus \{0\}$, $J_R(tu) \rightarrow -\infty$ as $t \rightarrow +\infty$ and the set \mathbb{R}^+u intersects \mathcal{N}_{J_R} at exactly one point, which is the unique globally maximum point of $J_R(tu)$ for $t > 0$.

Step 2. We prove that there exists $a_R > 0$ such that

$$\|u\| \geq a_R \text{ for any } u \in \mathcal{N}_{J_R}.$$

For any $u \in X_R \setminus \{0\}$, by (3.55) one has

$$\begin{aligned} \int_{B_R(0)} h(Lu)Lu &\geq \frac{1}{2} \int_{\{|Lu| \leq g(b_0)\}} |Lu|^2 + c \int_{\{|Lu| > g(b_0)\}} |Lu|^{(p+1)/p} \\ (3.57) \quad &\geq \frac{1}{2} |B_R(0)|^{\frac{1-p}{1+p}} \left(\int_{\{|Lu| \leq g(b_0)\}} |Lu|^{(p+1)/p} \right)^{2p/(p+1)} \\ &\quad + c \int_{\{|Lu| > g(b_0)\}} |Lu|^{(p+1)/p}. \end{aligned}$$

Moreover, by (H1), for any small $\gamma > 0$, there exist $c_\gamma > 0$ and $C > 0$ (independent of γ) such that

$$(3.58) \quad \int_{B_R(0)} f_k(u)u \leq \int_{B_R(0)} \gamma u^2 + c_\gamma |u|^{q+1} \leq C \|u\|^2 (\gamma + c_\gamma \|u\|^{q-1})$$

Here we used the embedding of X_R into $L^r(B_R(0))$ for $r = 2$ and $r = q + 1$. By choosing

$$\gamma = 2^{-\frac{4p+2}{p+1}} |B_R(0)|^{\frac{1-p}{1+p}} C^{-1},$$

and for any $u \in \mathcal{N}_{J_R}$, if $\|u\|^{q-1} \leq \gamma c_\gamma^{-1}$, by (3.57) and (3.58),

$$\begin{aligned} &\frac{1}{4} |B_R(0)|^{\frac{1-p}{1+p}} \left(\int_{\{|Lu| \leq g(b_0)\}} |Lu|^{(p+1)/p} \right)^{2p/(p+1)} + c \int_{\{|Lu| > g(b_0)\}} |Lu|^{(p+1)/p} \\ &\leq C \gamma 2^{2p/(p+1)} \left(\int_{\{|Lu| > g(b_0)\}} |Lu|^{(p+1)/p} \right)^{2p/(p+1)}. \end{aligned}$$

Since $u \neq 0$, we have $\int_{\{|Lu| > g(b_0)\}} |Lu|^{(p+1)/p} > 0$ and then

$$\int_{\{|Lu| > g(b_0)\}} |Lu|^{(p+1)/p} \geq \left(\frac{c}{C \gamma 2^{2p/(p+1)}} \right)^{\frac{p+1}{p-1}} > 0.$$

So that for any $u \in \mathcal{N}_{J_R}$ the following holds

$$\|u\| \geq \min \left\{ (\gamma c_\gamma^{-1})^{\frac{1}{q-1}}, \left(\frac{c}{C \gamma 2^{2p/(p+1)}} \right)^{\frac{p}{p-1}} \right\} := a_R > 0.$$

Step 3. We prove that there exists $b_R > 0$ such that $c_1(B_R(0)) \geq b_R$. Obviously, $c_1(B_R(0)) \geq 0$. Assume by contradiction that there exists $\{u_n\} \subset \mathcal{N}_{J_R}$ such that $J_R(u_n) \rightarrow 0$, as $n \rightarrow \infty$. We claim that $\{u_n\}$ is bounded in X_R . Indeed, if not we may assume $\|u_n\| \rightarrow \infty$, as $n \rightarrow \infty$. Let $v_n = u_n / \|u_n\|$ and assume that $v_n \rightharpoonup v$ weakly

in X_R . If $v = 0$, then by compactness of the embedding of X_R into $L^r(B_R(0))$ for $r = 2$ and $r = q + 1$, we get $\int_{B_R(0)} F_k(v_n) \rightarrow 0$, as $n \rightarrow \infty$. Then by Step 1,

$$J_R(u_n) = \max_{t \geq 0} J_R(tu_n) \geq J_R(v_n) = \int_{B_R(0)} H(Lv_n) + o_n(1).$$

Namely, $\int_{B_R(0)} H(Lv_n) = o_n(1)$. On the other hand, similar to (3.57),

$$\begin{aligned} \int_{B_R(0)} H(Lv_n) &\geq \frac{1}{2} |B_R(0)|^{\frac{1-p}{1+p}} \left(\int_{\{|Lv_n| \leq g(b_0)\}} |Lv_n|^{(p+1)/p} \right)^{2p/(p+1)} \\ &\quad + c \int_{\{|Lv_n| > g(b_0)\}} |Lv_n|^{(p+1)/p}. \end{aligned}$$

It follows that $v_n \rightarrow 0$ strongly in X_R , which contradicts the fact $\|v_n\| = 1$. So $v \neq 0$ and by (3.50), (3.55) and Fatou's Lemma,

$$o_n(1) = \frac{J_R(u_n)}{\|u_n\|^{\frac{p+1}{p}}} \leq C - \int_{B_R(0)} \frac{F_k(u_n)}{|u_n|^{(p+1)/p}} |v_n|^{(p+1)/p} \rightarrow -\infty.$$

This is a contradiction. Hence, $\{u_n\}$ is bounded in X_R . We may assume, up to a subsequence, $u_n \rightharpoonup u$ weakly in X_R and strongly in $L^2(B_R(0))$. Noting that $h(t)/t$ is strictly decreasing for $t > 0$, we have $0 < h(t)t \leq 2H(t)$ for all $t \neq 0$. Then by (H2),

$$\begin{aligned} o_n(1) &= J_R(u_n) - \frac{1}{2} \langle J'_R(u_n), u_n \rangle \\ &= \int_{B_R(0)} H(Lu_n) - \frac{1}{2} h(Lu_n) Lu_n + \frac{1}{2} \int_{B_R(0)} f_k(u_n) u_n - 2F_k(u_n) \\ &\geq \frac{1}{2} \int_{B_R(0)} f_k(u_n) u_n - 2F_k(u_n) \geq \frac{\theta - 2}{2} \int_{\{x \in B_R(0): |u_n| \leq a_0\}} F(u_n) \\ &\rightarrow \frac{\theta - 2}{2} \int_{\{x \in B_R(0): |u| \leq a_0\}} F(u), \text{ as } n \rightarrow \infty. \end{aligned}$$

It follows that

$$\int_{\{x \in B_R(0): |u| \leq a_0\}} F(u) = 0.$$

Since $u \in X_R$, from elliptic regularity we get $u \in C^{0,2/(p+1)}(\overline{B_R(0)})$, which yields $u = 0$. Analogously we get $\int_{B_R(0)} F_k(u_n) \rightarrow 0$, as $n \rightarrow \infty$ and

$$\int_{B_R(0)} H(Lu_n) = J_R(u_n) + o_n(1) = o_n(1).$$

Similar to (3.57),

$$\begin{aligned} \int_{B_R(0)} H(Lu_n) &\geq \frac{1}{2} |B_R(0)|^{\frac{1-p}{1+p}} \left(\int_{\{|Lu_n| \leq g(b_0)\}} |Lu_n|^{(p+1)/p} \right)^{2p/(p+1)} \\ &\quad + c \int_{\{|Lu_n| > g(b_0)\}} |Lu_n|^{(p+1)/p}. \end{aligned}$$

Thus $u_n \rightarrow 0$ strongly in X_R , which contradicts the fact $\|u\| \geq a_R$ for all $u \in \mathcal{N}_{J_R}$. \square

Define

$$\hat{m}_R : u \in X_R \setminus \{0\} \mapsto \hat{m}_R(u) \in \mathbb{R}^+ u \cap \mathcal{N}_{J_R}.$$

Similar as in [33], we have the following

Lemma 2.24. *There exists $\delta > 0$ such that $\|u\| \geq \delta$ for all $u \in \mathcal{N}_{J_R}$. In particular,*

$$\|\hat{m}_R(u)\| \geq \delta, \quad \text{for all } u \in X_R \setminus \{0\}.$$

Moreover, for each compact subset $\mathcal{W} \subset X_R \setminus \{0\}$, there exists a constant $C_{\mathcal{W}} > 0$ such that

$$\|\hat{m}_R(u)\| \leq C_{\mathcal{W}}, \quad \text{for all } u \in \mathcal{W}.$$

Proof. By (3.55), for any $u \in \mathcal{N}_{J_R}$, we have

$$b_1 \leq J_R(u) \leq \int_{B_R(0)} H(Lu) \leq C \|u\|^{p/(p+1)}.$$

Thus, there exists $\delta > 0$ such that $\|u\| \geq \delta$ for any $u \in \mathcal{N}_{J_R}$. Moreover, since $\hat{m}_R(u) = \hat{m}_R(u/\|u\|)$ for any $u \neq 0$, without loss generality, we may assume $\mathcal{W} \subset S_R := \{u \in X_R : \|u\| = 1\}$. In the following, we claim that there exists $C_{\mathcal{W}} > 0$ such that

$$(3.59) \quad J_R \leq 0 \text{ on } \mathbb{R}^+ u \setminus B_{C_{\mathcal{W}}}(0), \text{ for all } u \in \mathcal{W},$$

where $B_{C_{\mathcal{W}}}(0) = \{v \in X_R : \|v\| \leq C_{\mathcal{W}}\}$. If the claim (3.59) is true, then noting that $J_R(\hat{m}_R(u)) \geq b_1 > 0$ for all $0 \neq u \in X_R$, we have $\|\hat{m}_R(u)\| = \|\hat{m}_R(u/\|u\|)\| \leq C_{\mathcal{W}}$ for any $u \in \mathcal{W}$.

So let us prove (3.59). Assume by contradiction that there exists $\{u_n\} \subset \mathcal{W} \subset S_R$ with $u_n \rightarrow u$ strongly in \mathcal{W} and $\omega_n \in \mathbb{R}^+ u_n$ with $\omega_n = t_n u_n$, $t_n \rightarrow \infty$ such that $J_R(\omega_n) \geq 0$, as $n \rightarrow \infty$. For n large enough, by (3.55) one has

$$(3.60) \quad 0 \leq \frac{J_R(\omega_n)}{\|\omega_n\|^{(p+1)/p}} \leq C - \int_{B_R(0)} \frac{F_k(t_n u_n)}{|t_n u_n|^{(p+1)/p}} |u_n|^{(p+1)/p}.$$

Noting that $u_n \xrightarrow{a.e.} u \neq 0$, it follows from Fatou's Lemma and (3.60) that

$$\frac{J_R(\omega_n)}{\|\omega_n\|^{(p+1)/p}} \rightarrow -\infty$$

as $n \rightarrow \infty$, which is a contradiction. \square

Let $m_R := \hat{m}_R|_{S_R} : S_R \rightarrow \mathcal{N}_{J_R}$ and

$$K : S_R \rightarrow \mathbb{R}, \quad K(u) := J_R(m_R(u)), u \in S_R,$$

then \hat{m}_R is continuous and m_R is a homeomorphism between S_R and \mathcal{N}_{J_R} .

Proposition 2.25.

1) $K \in C^1(S_R, \mathbb{R})$ and $\langle K'(u), \omega \rangle = \|m_R(u)\| \langle J'_R(m_R(u)), \omega \rangle$, for all $\omega \in T_u(S_R)$;

2) If $\{\omega_n\} \subset S_R$ is a Palais-Smale sequence for K , then $\{m_R(\omega_n)\} \subset \mathcal{N}_{J_R}$ is a Palais-Smale sequence for J_R . Namely, if $K(\omega_n) \rightarrow d$ for some $d > 0$ and $\|K'(\omega_n)\|_* \rightarrow 0$, as $n \rightarrow \infty$, then $J_R(m_R(\omega_n)) \rightarrow d$ and $\|J'_R(m_R(\omega_n))\| \rightarrow 0$, as $n \rightarrow \infty$, where

$$\|K'(\omega_n)\|_* = \sup_{\substack{\phi \in T_{\omega_n}(S_R) \\ \|\phi\|=1}} \langle K'(\omega_n), \phi \rangle \quad \text{and} \quad \|J'_R(m_R(\omega_n))\| = \sup_{\substack{\phi \in X_R \\ \|\phi\|=1}} \langle J'_R(m_R(\omega_n)), \phi \rangle.$$

3) $\omega \in S_R$ is a critical point of K if and only if $m_R(\omega) \in \mathcal{N}_{J_R}$ is a critical point of J_R ;

4) $\inf_{S_R} K = \inf_{\mathcal{N}_{J_R}} J_R$.

Lemma 2.26. For any $R > 0$, $c_1(B_R(0)) \geq c_*(B_R(0))$.

Proof. Observing that S_R is a C^1 -manifold in X_R , by virtue of the Ekeland variational principle (see [20, Theorem 3.1]), there exists $\{u_n\} \subset \mathcal{N}_{J_R}$ such that

$$(3.61) \quad J_R(u_n) \rightarrow c_1(B_R(0)) > 0 \quad \text{and} \quad J'_R(u_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

It is standard to show that $\{u_n\}$ is bounded in X_R , thus up to a subsequence, $u_n \rightarrow u$ weakly in X_R , as $n \rightarrow \infty$. By means of the compactness of $X_R \hookrightarrow L^r(B_R(0))$ for any $r \geq (p+1)/p$, $u_n \rightarrow u$ strongly in $L^{q+1}(B_R(0))$. Then

$$(3.62) \quad \liminf_{n \rightarrow \infty} \int_{B_R(0)} h(Lu_n) Lu_n = \liminf_{n \rightarrow \infty} \int_{B_R(0)} f(u_n) u_n = \int_{B_R(0)} f(u) u.$$

By (3.55), we also have

$$\int_{B_R(0)} h(Lu_n) Lu_n \geq \frac{1}{2} \int_{|Lu_n| \leq g(b_0)} |Lu_n|^2 + c \int_{|Lu_n| > g(b_0)} |Lu_n|^{(p+1)/p}.$$

We claim that $u \not\equiv 0$. Indeed, otherwise by (3.62) we get

$$\lim_{n \rightarrow \infty} \int_{|Lu_n| \leq g(b_0)} |Lu_n|^2 = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{|Lu_n| > g(b_0)} |Lu_n|^{(p+1)/p} = 0.$$

Hence

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{B_R(0)} |Lu_n|^{(p+1)/p} &\leq \lim_{n \rightarrow \infty} \int_{|Lu_n| > g(b_0)} |Lu_n|^{(p+1)/p} \\ &+ \lim_{n \rightarrow \infty} \left(\int_{|Lu_n| \leq g(b_0)} |Lu_n|^2 \right)^{(p+1)/(2p)} |B_R(0)|^{(p-1)/(2p)} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, which implies $J_R(u_n) \rightarrow 0$, as $n \rightarrow \infty$. This is a contradiction.

Next let $u_0 = \hat{m}_R(u)u$ and $v_n = \hat{m}_R(u)u_n$. By (H7), H is convex. Therefore

$$\liminf_{n \rightarrow \infty} \int_{B_R(0)} H(Lv_n) \geq \int_{B_R(0)} H(Lu_0) \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{B_R(0)} F(v_n) = \int_{B_R(0)} F(u_0).$$

As $u_0 \in \mathcal{N}_{J_R}$ on the one hand on has

$$\liminf_{n \rightarrow \infty} J_R(v_n) \geq \int_{B_R(0)} H(Lu_0) - \int_{B_R(0)} F(u_0) \geq c_1(B_R(0)).$$

On the other hand, it follows from Lemma 2.23 and $u_n \in \mathcal{N}_{J_R}$ the following

$$\liminf_{n \rightarrow \infty} J_R(v_n) \leq \liminf_{n \rightarrow \infty} \max_{t \geq 0} J_R(tu_n) = \liminf_{n \rightarrow \infty} J_R(u_n) = c_1(B_R(0)).$$

and in turn $J_R(u_0) = c_1(B_R(0))$. By Proposition 3.61 $J'_R(u_0) = 0$ and by Proposition 2.20, (u_0, v_0) is a nontrivial critical point of I_R , namely $(u_0, v_0) \in \mathcal{N}_R$ where $v_0 = h(Lu_0)$. Finally,

$$c_*(B_R(0)) \leq I_R(u_0, v_0) = J_R(u_0) = c_1(B_R(0)).$$

□

Similar as in [4], one can prove the reversed inequality to get the following

Lemma 2.27. *For any $R > 0$,*

$$c_*(B_R(0)) = c_1(B_R(0)).$$

Lemma 2.28. *Let (u_R, v_R) be any ground state for the functional I_R , then $u_R v_R > 0$ in $B_R(0)$.*

Proof. Recalling that $\mathcal{S} = \mathcal{S}_k$, it is enough to prove $uv > 0$ in \mathbb{R}^2 for any $(u, v) \in \mathcal{S}_k$. For any $R > 0$ and any ground state (u_R, v_R) for the functional I_R , by Lemma 2.27 and Proposition 2.20, u_R is a ground state for the functional J_R . Let $\omega = L^{-1}(|Lu_R|)$, then $\omega > 0$ and $\omega \geq |u_R|$. Moreover, $\langle J'_R(t\omega), \omega \rangle = 0$, where $t = \hat{m}_R(\omega) > 0$. On the other hand,

$$\begin{aligned} c_1(B_R(0)) &\leq J_R(t\omega) = J_R(tu_R) + \int_{B_R(0)} F_k(t|u_R|) - F_k(t\omega) \\ &\leq c_1(B_R(0)) + \int_{B_R(0)} F_k(t|u_R|) - F_k(t\omega). \end{aligned}$$

So that $\int_{B_R(0)} F_k(t|u_R|) - F_k(t\omega) \geq 0$. It follows from (H7) that $|u_R| = \omega > 0$. If $u_R > 0$ in $B_R(0)$, then by means of the maximum principle, $v_R > 0$ in $B_R(0)$ and $u_R v_R > 0$ in $B_R(0)$. Similarly, if $u_R < 0$ in $B_R(0)$, $u_R v_R > 0$ in $B_R(0)$. □

As a consequence of Lemma 2.27 and Lemma 2.28, see also [4, Remark 4.11], we have

Lemma 2.29. *The map $R \mapsto c_*(B_R(0))$ is decreasing for $R > 0$.*

Lemma 2.30. *For any $R > 0$, we have $c_*(B_R(0)) \geq c_*(\mathbb{R}^2)$.*

Proof. For any $R > 0$, let $z_R = (u_R, v_R)$ be a ground state solution of I_R . Namely, $I_R(z_R) = c_*(B_R(0))$ and $I'_R(z_R) = 0$. We extend $z_R \in E_R$ to $z_R \in E$ by zero extension outside $B_R(0)$. Then, as in Lemma 2.15, $\{z_R\}$ turns out to be bounded in E . Up to a subsequence, we may assume $z_R \rightharpoonup z_0$ weakly in E , as $R \rightarrow \infty$, then $z_0 = (u_0, v_0) \in E$ is a nonnegative solution to (3.51), namely $\Phi'_k(z_0) = 0$.

If $z_0 \neq 0$, by (H2) and Fatou's Lemma, we have for any $r \leq R$,

$$\begin{aligned}
c_*(B_r(0)) &\geq \lim_{R \rightarrow \infty} c_*(B_R(0)) = \lim_{R \rightarrow \infty} \left(I_R(z_R) - \frac{1}{2} \langle I'_R(z_R), z_R \rangle \right) \\
&= \lim_{R \rightarrow \infty} \left(\int_{B_R(0)} \frac{1}{2} f_k(u_R) u_R - F_k(u_R) + \int_{B_R(0)} \frac{1}{2} g_k(v_R) v_R - G_k(v_R) \right) \\
&\geq \int_{\mathbb{R}^2} \frac{1}{2} f_k(u_0) u_0 - F_k(u_0) + \int_{\mathbb{R}^2} \frac{1}{2} g_k(v_0) v_0 - G_k(v_0) \\
&= \Phi_k(z_0) - \frac{1}{2} \langle \Phi'_k(z_0), z_0 \rangle = \Phi_k(z_0) \geq c_*(\mathbb{R}^2).
\end{aligned}$$

If $z_0 = 0$, then $\{z_R\}$ satisfies one of the following alternatives:

(1) (*Vanishing*)

$$\lim_{R \rightarrow \infty} \sup_{y \in \mathbb{R}^2} \int_{B_r(y)} (u_R^2 + v_R^2) dx = 0, \quad \text{for all } r > 0;$$

(2) (*Nonvanishing*) there exist $\nu > 0$, $r_0 > 0$ and $\{y_R\} \subset \mathbb{R}^2$ such that

$$\lim_{R \rightarrow \infty} \int_{B_{r_0}(y_R)} (u_R^2 + v_R^2) dx \geq \nu.$$

As in Proposition 2.4 *Vanishing* does not occur. So let $\tilde{u}_R := u_R(\cdot + y_R)$ and $\tilde{v}_R := v_R(\cdot + y_R)$, then $\tilde{z}_R = (\tilde{u}_R, \tilde{v}_R)$ is bounded in $H^1(\mathbb{R}^2)$ and $\tilde{z}_R \rightharpoonup \tilde{z}_0 \neq 0$ weakly in $H^1(\mathbb{R}^2)$. Moreover, let $\tilde{z}_0 = (\tilde{u}_0, \tilde{v}_0)$, we know \tilde{u}_0, \tilde{v}_0 are nonnegative. Obviously, $|y_R| \leq R + r_0$. Assume that, up to a rotation, $y_R/|y_R| \rightarrow (0, -1) \in \mathbb{R}^2$ and $(\tilde{u}_0, \tilde{v}_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ satisfies

$$(3.63) \quad \begin{cases} -\Delta \tilde{u}_0 + V_0 \tilde{u}_0 = g_k(\tilde{v}_0) \\ -\Delta \tilde{v}_0 + V_0 \tilde{v}_0 = f_k(\tilde{u}_0) \end{cases}$$

where $\Omega = \mathbb{R}^2$ or $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > d\}$, where $d := \liminf_{R \rightarrow \infty} \text{dist}(y_R, \partial B_R(0))$. If $\Omega = \mathbb{R}^2$ the proof follows. If $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > d\}$, then by the Hopf Lemma, $\partial \tilde{u}_0 / \partial \eta < 0$ and $\partial \tilde{v}_0 / \partial \eta < 0$ on $\partial \Omega$, where η is the outward pointing unit normal to $\partial \Omega$. Finally from the Pohozaev type identity proved in [19, Proposition 1.2] (see also [27, Lemma 3.1]) one actually has

$$\int_{\partial \Omega} \frac{\partial \tilde{u}_0}{\partial n} \frac{\partial \tilde{v}_0}{\partial n} = 0,$$

which is a contradiction. □

Proof of Theorem 2.2 completed. Thanks to Lemma 2.30 any ground state solution (u, v) to (2.1) does not change sign. Assume $u > 0$ and $v > 0$ in \mathbb{R}^2 . Setting

$$f_1(u, v) = g(v) - V_0 u \quad \text{and} \quad f_2(u, v) = f(u) - V_0 v,$$

as a consequence of [7, Theorem 1] and (H1), (u, v) is radially symmetric and strictly decreasing with respect to the same point, which we denote by x_0 . Clearly, $\Delta u(x_0) \leq 0$ and $\Delta v(x_0) \leq 0$. To complete the proof of Theorem 2.2, we next prove that actually

$\Delta u(x_0) < 0$ and $\Delta v(x_0) < 0$. Indeed, if not, without loss of generality we may assume $\Delta u(x_0) = 0$ and then $g(v(x_0)) = V_0 u(x_0)$. Let $u_1 = u - u(x_0)$, then $u_1(x) \leq 0$ in \mathbb{R}^2 and

$$\begin{aligned} -\Delta u_1 &= -\Delta u = g(v) - V_0 u \\ &\leq g(v(x_0)) - V_0 u(x_0) - V_0 u_1 \\ &= -V_0 u_1. \end{aligned}$$

Namely, $-\Delta u_1 + V_0 u_1 \leq 0$ in \mathbb{R}^2 . Noting that $u_1(0) = 0$, by the maximum principle, $u_1 \equiv 0$ in \mathbb{R}^2 , which is a contradiction. Therefore, $\Delta u(x_0) < 0$. Similarly, one has $\Delta v(x_0) < 0$ as well. Finally, by Proposition 2.10, $u(x + x_z), v(x + x_z) \rightarrow 0$, as $|x| \rightarrow \infty$ uniformly for any $z = (u, v) \in \mathcal{S}$. Since u, v do not change the sign, using the maximum principle, we conclude that there exist $C, c > 0$, independent of $z = (u, v) \in \mathcal{S}$, such that

$$|D^\alpha u(x)| + |D^\alpha v(x)| \leq C \exp(-c|x - x_0|), \quad x \in \mathbb{R}^2, \quad |\alpha| = 0, 1$$

□

3. PROOF OF THEOREM 1.2

3.1. Functional setting. By setting $u(x) = \varphi(\varepsilon x), v(x) = \psi(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$, (1.1) is equivalent to

$$(5.1) \quad \begin{cases} -\Delta u + V_\varepsilon(x)u = g(v) \\ -\Delta v + V_\varepsilon(x)v = f(u) \end{cases}$$

We next consider (5.1). Let H_ε be the completion of $C_0^\infty(\mathbb{R}^2)$ with respect to the inner product

$$(u, v)_{1, \varepsilon} := \int_{\mathbb{R}^2} \nabla u \nabla v + V_\varepsilon(x)uv$$

and the norm

$$\|u\|_{1, \varepsilon}^2 := (u, u)_{1, \varepsilon}, \quad u, v \in H_\varepsilon.$$

Let $E_\varepsilon := H_\varepsilon \times H_\varepsilon$ with the inner product

$$(z_1, z_2)_\varepsilon := (u_1, u_2)_{1, \varepsilon} + (v_1, v_2)_{1, \varepsilon}, \quad z_i = (u_i, v_i) \in E_\varepsilon, \quad i = 1, 2.$$

and the norm $\|z\|_\varepsilon^2 = \|(u, v)\|_\varepsilon^2 = \|u\|_{1, \varepsilon}^2 + \|v\|_{1, \varepsilon}^2$. We have the orthogonal space decomposition $E_\varepsilon = E_\varepsilon^+ \oplus E_\varepsilon^-$, where

$$E_\varepsilon^+ := \{(u, u) \mid u \in H_\varepsilon\} \quad \text{and} \quad E_\varepsilon^- := \{(u, -u) \mid u \in H_\varepsilon\}.$$

For each $z = (u, v) \in E_\varepsilon$,

$$z = z^+ + z^- = ((u + v)/2, (u + v)/2) + ((u - v)/2, (v - u)/2).$$

Weak solutions of (5.1) are critical points of the associated energy functional

$$\Phi_\varepsilon(z) := \int_{\mathbb{R}^2} \nabla u \nabla v + V_\varepsilon(x)uv - I(z), \quad z = (u, v) \in E_\varepsilon,$$

where $I(z) = \int_{\mathbb{R}^2} F(u) + G(v)$. Then $\Phi_\varepsilon \in C^1(E, \mathbb{R})$ and

$$\langle \Phi'_\varepsilon(z), w \rangle = \int_{\mathbb{R}^2} (\nabla u \nabla w_2 + \nabla v \nabla w_1 + V_\varepsilon(x)uw_2 + V_\varepsilon(x)vw_1) - \int_{\mathbb{R}^2} (f(u)w_1 + g(v)w_2),$$

for all $z = (u, v), w = (w_1, w_2) \in E_\varepsilon$. Moreover, Φ_ε can be rewritten as follows

$$(5.2) \quad \Phi_\varepsilon(z) := \frac{1}{2}\|z^+\|_\varepsilon^2 - \frac{1}{2}\|z^-\|_\varepsilon^2 - I(z).$$

We know that if $z \in E_\varepsilon$ is a nontrivial critical point of Φ_ε , then $z \in E_\varepsilon \setminus E_\varepsilon^-$. In the spirit of [33], we define the generalized Nehari Manifold

$$\mathcal{N}_\varepsilon := \{z \in E_\varepsilon \setminus E_\varepsilon^- : \langle \Phi'_\varepsilon(z), z \rangle_\varepsilon = 0, \langle \Phi'_\varepsilon(z), \varphi \rangle_\varepsilon = 0 \text{ for all } \varphi \in E_\varepsilon^-\}.$$

Let

$$c_\varepsilon := \inf_{z \in \mathcal{N}_\varepsilon} \Phi_\varepsilon(z),$$

then c_ε is the least energy for system (5.1), the so-called ground state level.

For $z \in E_\varepsilon \setminus E_\varepsilon^-$, set

$$\hat{E}_\varepsilon(z) = E_\varepsilon^- \oplus \mathbb{R}^+ z = E_\varepsilon^- \oplus \mathbb{R}^+ z^+,$$

where $\mathbb{R}^+ z := \{tz : t \geq 0\}$. From [16, 33, 34] we have the following properties of \mathcal{N}_ε , which will be used later.

Lemma 3.1. *Under the assumptions in Theorem 1.2, we have:*

- 1) for any $z \in \mathcal{N}_\varepsilon$, $\Phi_\varepsilon|_{\hat{E}_\varepsilon(z)}$ admits a unique maximum point which occurs precisely at z ;
- 2) for any $z \in E_\varepsilon \setminus E_\varepsilon^-$, the set $\hat{E}_\varepsilon(z)$ intersects \mathcal{N}_ε at exactly one point $\hat{m}_\varepsilon(z)$, which is the unique global maximum point of $\Phi_\varepsilon|_{\hat{E}_\varepsilon(z)}$.

3.2. Lower and upper bounds for c_ε .

Proposition 3.2. *There exists $c_0 > 0$ (independent of ε) such that for $\varepsilon > 0$ sufficiently small,*

$$c_\varepsilon = \inf_{z \in E_\varepsilon \setminus E_\varepsilon^-} \max_{\omega \in \hat{E}_\varepsilon(z)} \Phi_\varepsilon(\omega) \in (c_0, 4\pi/\alpha_0).$$

Proof. The min-max characterization is standard and we refer to [16]. Here we are concerned with estimating from below and above the critical level C_ε .

Lower bound. On one hand, for any $z \in E_\varepsilon$, we know $\hat{E}_\varepsilon(z) = \hat{E}_\varepsilon(z^+)$. Then, for any $a > 0$

$$\begin{aligned} c_\varepsilon &= \inf_{z \in E_\varepsilon \setminus E_\varepsilon^-} \max_{\omega \in \hat{E}_\varepsilon(z)} \Phi_\varepsilon(\omega) = \inf_{z \in E_\varepsilon^+ \setminus \{0\}} \max_{\omega \in \hat{E}_\varepsilon(z)} \Phi_\varepsilon(\omega) \\ &= \inf_{z \in S_{a,\varepsilon}^+} \max_{\omega \in \hat{E}_\varepsilon(z)} \Phi_\varepsilon(\omega) \geq \inf_{z \in S_{a,\varepsilon}^+} \max_{\omega \in \mathbb{R}^+ z} \Phi_\varepsilon(\omega), \end{aligned}$$

where $S_{a,\varepsilon}^+ := \{z \in E_\varepsilon^+ : \|z\|_\varepsilon = a\}$. On the other hand, recalling that f, g have critical growth with critical exponent α_0 , by (H1), for some $\alpha' > \alpha_0$, there exists $C > 0$ such that

$$(5.3) \quad F(t) \leq \frac{1}{4}V_0|t|^2 + C|t|^3 \left(e^{\alpha' t^2} - 1\right), \quad G(t) \leq \frac{1}{4}V_0|t|^2 + C|t|^3 \left(e^{\alpha' t^2} - 1\right), \quad t \in \mathbb{R}.$$

By the Pohozaev-Trudinger-Moser inequality, there exists $a > 0$ sufficiently small such that

$$\int_{\mathbb{R}^2} \left(e^{2\alpha' u^2} - 1\right) \leq 1,$$

for any $u \in H^1(\mathbb{R}^2)$ with $\|u\|_{H^1} \leq a$. Then, for any $z = (u, u) \in S_{a,\varepsilon}^+$,

$$\begin{aligned} \max_{\omega \in \mathbb{R}^{+z}} \Phi_\varepsilon(\omega) &\geq \Phi_\varepsilon(z) = \int_{\mathbb{R}^2} |\nabla u|^2 + V_\varepsilon(x)u^2 - \int_{\mathbb{R}^2} F(u) + G(u) \\ &\geq \|u\|_{1,\varepsilon}^2 - V_0/2 \int_{\Omega} u^2 - 2C \int_{\mathbb{R}^2} |u|^3 \left(e^{\alpha' u^2} - 1 \right) \\ &\geq C' \|u\|_{1,\varepsilon}^2 - 2C \left(\int_{\Omega} u^6 \right)^{1/2} \geq \|u\|_{1,\varepsilon}^2 (C' - 2CC_6^3 \|u\|_{1,\varepsilon}), \end{aligned}$$

where $C' = \min\{1, V_0\}/2$ and C_6 is the Sobolev's constant of the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^6(\mathbb{R}^2)$. Thus, taking $a > 0$ fixed but small enough, for any $z = (u, u) \in S_{a,\varepsilon}^+$, we have $\|u\|_{1,\varepsilon}^2 = a^2/2$ and

$$\max_{\omega \in \mathbb{R}^{+z}} \Phi_\varepsilon(\omega) \geq \|u\|_{1,\varepsilon}^2 [C' - 2CC_6^3 \|u\|_{1,\varepsilon}] \geq a^2/6 > 0.$$

Thus, for any $\varepsilon > 0$, $c_\varepsilon \geq c_0 = a^2/6$.

Upper bound. By (H5) and $V(0) = V_0$, for some fixed $r > 0$ and $\varepsilon_0 > 0$ such that

$$(5.4) \quad \beta_0 > \frac{4e^{\frac{r^2}{2} \max_{|x| \leq \varepsilon r} V(x)}}{\alpha_0 r^2}, \quad \varepsilon \in (0, \varepsilon_0),$$

we consider the following so-called Moser sequence

$$(5.5) \quad \omega_k(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log k)^{1/2}, & |x| \leq r/k; \\ \frac{\log \frac{r}{|x|}}{(\log k)^{1/2}}, & r/k \leq |x| \leq r; \\ 0, & |x| \geq r. \end{cases}$$

Then, one easily checks that $\|\nabla \omega_k\|_2 = 1$ and $\|\omega_k\|_2^2 = r^2/(4 \log k) + o(r^2/\log k)$. Let $d_k(r) := r^2/4 + o_k(1)$ where $o_k(1) \rightarrow 0$, as $k \rightarrow +\infty$ and $\tilde{\omega}_{k,\varepsilon} := \omega_k/\|\omega_k\|_{1,\varepsilon}$, then $\|\tilde{\omega}_{k,\varepsilon}\|_{1,\varepsilon} = 1$ and for k large enough,

$$(5.6) \quad \tilde{\omega}_{k,\varepsilon}^2(x) \geq \frac{1}{2\pi} \left(\log k - d_{k,\varepsilon}(r) \right) \quad \text{for } |x| \leq \frac{r}{k},$$

where $d_{k,\varepsilon}(r) = d_k(r) \max_{|x| \leq \varepsilon r} V(x) \geq V_0 d_k(r)$.

Suppose by contradiction that for some fixed $\varepsilon \in (0, \varepsilon_0)$ and for all k ,

$$\sup_{z \in \hat{E}((\tilde{\omega}_{k,\varepsilon}, \tilde{\omega}_{k,\varepsilon}))} \Phi_\varepsilon(z) \geq 4\pi/\alpha_0.$$

Then $\Phi_\varepsilon(\hat{m}((\tilde{\omega}_{k,\varepsilon}, \tilde{\omega}_{k,\varepsilon}))) \geq 4\pi/\alpha_0$ for all k , where $\hat{m}((\tilde{\omega}_{k,\varepsilon}, \tilde{\omega}_{k,\varepsilon})) \in \mathcal{N}_\varepsilon$ and

$$\hat{m}((\tilde{\omega}_{k,\varepsilon}, \tilde{\omega}_{k,\varepsilon})) = \tau_k(\tilde{\omega}_{k,\varepsilon}, \tilde{\omega}_{k,\varepsilon}) + (u_k, -u_k) \in \hat{E}((\tilde{\omega}_{k,\varepsilon}, \tilde{\omega}_{k,\varepsilon})).$$

Namely,

$$(5.7) \quad \tau_k^2 - \int_{\mathbb{R}^2} (|\nabla u_k|^2 + V_\varepsilon(x)u_k^2) - \int_{\mathbb{R}^2} [F(\tau_k \tilde{\omega}_{k,\varepsilon} + u_k) + G(\tau_k \tilde{\omega}_{k,\varepsilon} - u_k)] \geq 4\pi/\alpha_0$$

and

(5.8)

$$\tau_k^2 - \int_{\mathbb{R}^2} (|\nabla u_k|^2 + V_\varepsilon(x)u_k^2) = \int_{\mathbb{R}^2} [f(\tau_k\tilde{\omega}_{k,\varepsilon} + u_k)(\tau_k\tilde{\omega}_{k,\varepsilon} + u_k) + g(\tau_k\tilde{\omega}_{k,\varepsilon} - u_k)(\tau_k\tilde{\omega}_{k,\varepsilon} - u_k)].$$

Claim: $\lim_{k \rightarrow \infty} \tau_k = 4\pi/\alpha_0$. Indeed, from (5.7), we get $\tau_k^2 \geq 4\pi/\alpha_0$. From (H5), given $\rho > 0$, there exists R_ρ such that

$$tf(t) \geq (\beta_0 - \rho)e^{\alpha_0 t^2} \text{ for all } t \geq R_\rho.$$

and the same holds true also for $tg(t)$. Noting that

$$\tau_k\tilde{\omega}_{k,\varepsilon} = \frac{\tau_k}{\|\omega_k\|_\varepsilon} \frac{\sqrt{\log k}}{\sqrt{2\pi}} \rightarrow +\infty, \text{ as } k \rightarrow \infty, \quad x \in B_{r/k},$$

by choosing k sufficiently large, we get $\max\{\tau_k\tilde{\omega}_{k,\varepsilon} + u_k, \tau_k\tilde{\omega}_{k,\varepsilon} - u_k\} \geq R_\rho$ for all $x \in B_{r/k}$. So that by (5.6),

$$\begin{aligned} \tau_k^2 &\geq \int_{B_{r/k}} [f(\tau_k\tilde{\omega}_{k,\varepsilon} + u_k)(\tau_k\tilde{\omega}_{k,\varepsilon} + u_k) + g(\tau_k\tilde{\omega}_{k,\varepsilon} - u_k)(\tau_k\tilde{\omega}_{k,\varepsilon} - u_k)] \\ &\geq (\beta_0 - \rho) \int_{B_{r/k}} e^{\alpha_0(\tau_k\tilde{\omega}_{k,\varepsilon})^2} dx \\ (5.9) \quad &\geq \pi r^2 (\beta_0 - \rho) e^{\frac{\alpha_0}{2\pi} \tau_k^2 [\log k - d_{k,\varepsilon}(r)] - 2 \log k}, \end{aligned}$$

which implies that $\{\tau_k\}$ is bounded. By (5.9), as a consequence of the boundedness of $\{\tau_k\}$, we know $\limsup_{k \rightarrow \infty} \tau_k^2 \leq 4\pi/\alpha_0$. In fact, if not we have

$$\limsup_{k \rightarrow \infty} e^{\frac{\alpha_0}{2\pi} \tau_k^2 [\log k - d_{k,\varepsilon}(r)] - 2 \log k} = \infty,$$

which is a contradiction, and the claim is proved.

As $\omega_k \rightarrow 0$ a.e. in \mathbb{R}^2 , by the Lebesgue dominated convergence theorem

$$\int_{\{x \in B_r : \tau_k\tilde{\omega}_{k,\varepsilon} < R_\rho\}} \min\{f(\tau_k\tilde{\omega}_{k,\varepsilon})\tau_k\tilde{\omega}_{k,\varepsilon}, g(\tau_k\tilde{\omega}_{k,\varepsilon})\tau_k\tilde{\omega}_{k,\varepsilon}\} dx \rightarrow 0, \quad k \rightarrow \infty$$

and

$$\int_{\{x \in B_r : \tau_k\tilde{\omega}_{k,\varepsilon} < R_\rho\}} e^{\alpha_0(\tau_k\tilde{\omega}_{k,\varepsilon})^2} dx \rightarrow \pi r^2.$$

Then, from (5.8) and (H4) we have obtain

$$\begin{aligned} \tau_k^2 &\geq \int_{B_r} [f(\tau_k\tilde{\omega}_{k,\varepsilon} + u_k)(\tau_k\tilde{\omega}_{k,\varepsilon} + u_k) + g(\tau_k\tilde{\omega}_{k,\varepsilon} - u_k)(\tau_k\tilde{\omega}_{k,\varepsilon} - u_k)] dx \\ &\geq (\beta_0 - \rho) \int_{B_r} e^{\alpha_0(\tau_k\tilde{\omega}_{k,\varepsilon})^2} dx - (\beta_0 - \rho) \int_{\{x \in B_r : \tau_k\tilde{\omega}_{k,\varepsilon} < R_\rho\}} e^{\alpha_0(\tau_k\tilde{\omega}_{k,\varepsilon})^2} dx \\ &\quad + \int_{\{x \in B_r : \tau_k\tilde{\omega}_{k,\varepsilon} < R_\rho\}} \min\{f(\tau_k\tilde{\omega}_{k,\varepsilon})\tau_k\tilde{\omega}_{k,\varepsilon}, g(\tau_k\tilde{\omega}_{k,\varepsilon})\tau_k\tilde{\omega}_{k,\varepsilon}\} dx \\ &= (\beta_0 - \rho) \left[\int_{B_r} e^{\alpha_0(\tau_k\tilde{\omega}_{k,\varepsilon})^2} dx - \pi r^2 \right]. \end{aligned}$$

In the following, we estimate the term $\int_{B_r} e^{\alpha_0(\tau_k \tilde{\omega}_{k,\varepsilon})^2} dx$. Observe first that from (5.6) one has

$$\int_{B_{r/k}} e^{\alpha_0(\tau_k \tilde{\omega}_{k,\varepsilon})^2} dx \geq \pi r^2 e^{\frac{\alpha_0}{2\pi} \tau_k^2 [\log k - d_{k,\varepsilon}(r)] - 2 \log k}.$$

Noting that $\tau_k^2 \geq 4\pi/\alpha_0$ and $\tau_k^2 \rightarrow 4\pi/\alpha_0$, we have

$$\liminf_{k \rightarrow \infty} \int_{B_{r/k}} e^{\alpha_0(\tau_k \tilde{\omega}_{k,\varepsilon})^2} dx \geq \pi r^2 e^{-\max_{|x| \leq \varepsilon r} V(x) r^2 / 2}.$$

Secondly, by using the change of variable $s = r e^{-\|\omega_k\|_\varepsilon \sqrt{\log k} t}$, one has

$$\begin{aligned} \int_{B_r \setminus B_{r/k}} e^{4\pi(\tilde{\omega}_{k,\varepsilon})^2} dx &= 2\pi r^2 \|\omega_k\|_\varepsilon \sqrt{\log k} \int_0^{\frac{\sqrt{\log k}}{\|\omega_k\|_\varepsilon}} e^{2(t^2 - \|\omega_k\|_\varepsilon \sqrt{\log k} t)} dt \\ &\geq 2\pi r^2 \|\omega_k\|_\varepsilon \sqrt{\log k} \int_0^{\frac{\sqrt{\log k}}{\|\omega_k\|_\varepsilon}} e^{-2\|\omega_k\|_\varepsilon \sqrt{\log k} t} dt \\ &= \pi r^2 (1 - e^{-2 \log k}). \end{aligned}$$

Thus

$$\liminf_{k \rightarrow \infty} \int_{B_r} e^{\alpha_0(\tau_k \tilde{\omega}_{k,\varepsilon})^2} dx \geq \pi r^2 (e^{-\max_{|x| \leq \varepsilon r} V(x) r^2 / 2} + 1),$$

which implies

$$4\pi/\alpha_0 = \lim_{k \rightarrow +\infty} \tau_k^2 \geq (\beta_0 - \rho) \pi r^2 e^{-\max_{|x| \leq \varepsilon r} V(x) r^2 / 2}.$$

As ρ is arbitrary, we have

$$\beta_0 \leq \frac{4e^{\frac{r^2}{2} \max_{|x| \leq \varepsilon r} V(x)}}{\alpha_0 r^2},$$

which contradicts (5.4). Therefore, $c_\varepsilon < 4\pi/\alpha_0$ for $\varepsilon \in (0, \varepsilon_0)$. \square

3.3. Existence of solutions to system (5.1).

Let us define

$$\hat{m}_\varepsilon : z \in E_\varepsilon \setminus E_\varepsilon^- \mapsto \hat{m}_\varepsilon(z) \in \hat{E}_\varepsilon(z) \cap \mathcal{N}_\varepsilon.$$

Lemma 3.3. *There exists $\delta > 0$ (independent of ε) such that $\|z^+\|_\varepsilon \geq \delta$ for all $z \in \mathcal{N}_\varepsilon$. In particular,*

$$\|\hat{m}_\varepsilon(z)^+\|_\varepsilon \geq \delta \quad \text{for all } z \in E_\varepsilon \setminus E_\varepsilon^-.$$

Moreover, for each compact subset $\mathcal{W} \subset E_\varepsilon \setminus E_\varepsilon^-$, there exists a constant $C_{\mathcal{W},\varepsilon} > 0$ such that

$$\|\hat{m}_\varepsilon(z)\|_\varepsilon \leq C_{\mathcal{W},\varepsilon} \quad \text{for all } z \in \mathcal{W}.$$

Let

$$S_\varepsilon^+ := \{z \in E_\varepsilon^+ : \|z\|_\varepsilon = 1\},$$

then S_ε^+ is a C^1 -submanifold of E_ε^+ and the tangent manifold of S_ε^+ at $z \in S_\varepsilon^+$ is

$$T_z(S_\varepsilon^+) = \{\omega \in E_\varepsilon^+ : (\omega, z)_\varepsilon = 0\}.$$

Let

$$m_\varepsilon := \hat{m}_\varepsilon|_{S_\varepsilon^+} : S_\varepsilon^+ \longrightarrow \mathcal{N}_\varepsilon,$$

then by Lemma 3.3, \hat{m}_ε is continuous and m_ε is a homeomorphism between S_ε^+ and \mathcal{N}_ε . Define

$$\Psi_\varepsilon : S_\varepsilon^+ \longrightarrow \mathbb{R}, \quad \Psi_\varepsilon(z) := \Phi_\varepsilon(m_\varepsilon(z)), \quad z \in S_\varepsilon^+,$$

then, as a consequence of [34, Corollary 4.3], for any fixed $\varepsilon > 0$, we have the following

Proposition 3.4.

1) $\Psi_\varepsilon \in C^1(S_\varepsilon^+, \mathbb{R})$ and

$$\langle \Psi'_\varepsilon(z), \omega \rangle_\varepsilon = \|m_\varepsilon(z)^+\| \langle \Phi'_\varepsilon(m_\varepsilon(z)), \omega \rangle_\varepsilon \quad \text{for all } \omega \in T_z(S_\varepsilon^+);$$

2) If $\{\omega_n\} \subset S_\varepsilon^+$ is a Palais-Smale sequence for Ψ_ε , then $\{m_\varepsilon(\omega_n)\} \subset \mathcal{N}_\varepsilon$ is a Palais-Smale sequence for Φ_ε . Namely, if $\Psi_\varepsilon(\omega_n) \rightarrow d$ for some $d > 0$ and $\|\Psi'_\varepsilon(\omega_n)\|_* \rightarrow 0$, as $n \rightarrow \infty$, then $\Phi_\varepsilon(m_\varepsilon(\omega_n)) \rightarrow d$ and $\|\Phi'_\varepsilon(m_\varepsilon(\omega_n))\| \rightarrow 0$, as $n \rightarrow \infty$, where

$$\|\Psi'_\varepsilon(\omega_n)\|_* = \sup_{\substack{\phi \in T_{\omega_n}(S_\varepsilon^+) \\ \|\phi\|_\varepsilon = 1}} \langle \Psi'_\varepsilon(\omega_n), \phi \rangle_\varepsilon \quad \text{and} \quad \|\Phi'_\varepsilon(m_\varepsilon(\omega_n))\| = \sup_{\substack{\phi \in E_\varepsilon \\ \|\phi\|_\varepsilon = 1}} \langle \Phi'_\varepsilon(m_\varepsilon(\omega_n)), \phi \rangle_\varepsilon;$$

3) $\omega \in S_\varepsilon^+$ is a critical point of Ψ_ε if and only if $m_\varepsilon(\omega) \in \mathcal{N}_\varepsilon$ is a critical point of Φ_ε ;

4) $\inf_{S_\varepsilon^+} \Psi_\varepsilon = \inf_{\mathcal{N}_\varepsilon} \Phi_\varepsilon$.

Since S_ε^+ is a regular C^1 -submanifold of E_ε^+ , by Proposition 3.2 and Proposition 3.4, it follows from the Ekeland variational principle (see [20, Theorem 3.1]) that there exists $\{\omega_n\} \subset S_\varepsilon^+$ such that

$$\Psi_\varepsilon(\omega_n) \rightarrow c_\varepsilon > 0 \quad \text{and} \quad \|\Psi'_\varepsilon(\omega_n)\|_* \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Let $z_n = m(\omega_n) \in \mathcal{N}_\varepsilon$, then

$$(5.10) \quad \Phi_\varepsilon(z_n) \rightarrow c_\varepsilon > 0 \quad \text{and} \quad \|\Phi'_\varepsilon(z_n)\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Similar as in [16], one has the following two propositions:

Proposition 3.5. *There exists C (independent of ε) such that for all $\varepsilon > 0$ and $n \in \mathbb{N}$:*

- 1) $\|z_n\|_\varepsilon = \|(u_n, v_n)\|_\varepsilon \leq C(1 + c_\varepsilon)$;
- 2) $\int_{\mathbb{R}^2} f(u_n)u_n \, dx \leq C(1 + c_\varepsilon)$ and $\int_{\mathbb{R}^2} g(v_n)v_n \, dx \leq C(1 + c_\varepsilon)$;
- 3) $\int_{\mathbb{R}^2} F(u_n) \, dx \leq C(1 + c_\varepsilon)$ and $\int_{\mathbb{R}^2} G(v_n) \, dx \leq C(1 + c_\varepsilon)$.

Up to a subsequence, there exists $z_\varepsilon = (u_\varepsilon, v_\varepsilon) \in E_\varepsilon$ such that $z_n \rightharpoonup z_\varepsilon$ in E_ε and $z_n \xrightarrow{a.e.} z_\varepsilon$ in \mathbb{R}^2 , as $n \rightarrow \infty$, which is actually a weak solution to (5.1), precisely we have

Proposition 3.6. *The weak limit z_ε is a critical point of Φ_ε .*

3.4. Asymptotic behavior of c_ε . By Proposition 3.6, it suffices to show $z_\varepsilon \neq 0$. For this purpose, in the following, we investigate the relation between c_* and c_ε , where c_*, c_ε are the corresponding least energies to System (2.1) and (5.1) respectively.

Lemma 3.7. *With the assumptions of Theorem 1.2, we have*

$$\limsup_{\varepsilon \rightarrow 0} c_\varepsilon \leq c_*.$$

Proof. By Theorem A, there exists $z = (u, v) \in \mathcal{N}$ such that

$$c_* = \max_{\omega \in \hat{E}(z)} \Phi(\omega) = \max_{\omega \in \hat{E}(z^+)} \Phi(\omega).$$

Noting that $z \in E \setminus E^-$, we know for any $\varepsilon > 0$, $z \in E_\varepsilon \setminus E_\varepsilon^-$. Then, by Lemma 3.1, for any $\varepsilon > 0$

$$c_\varepsilon \leq \max_{\omega \in \hat{E}_\varepsilon(z)} \Phi_\varepsilon(\omega) = \Phi_\varepsilon(\hat{m}_\varepsilon(z)).$$

Recalling that $\hat{m}_\varepsilon(z) \in \hat{E}_\varepsilon(z) \cap \mathcal{N}_\varepsilon$, there exist $s_\varepsilon \geq 0$, $t_\varepsilon \in \mathbb{R}$ and $\varphi_\varepsilon \in H_\varepsilon$, $\|\varphi_\varepsilon\|_\varepsilon = 1$ such that $\hat{m}_\varepsilon(z) = s_\varepsilon z + t_\varepsilon(\varphi_\varepsilon, -\varphi_\varepsilon)$.

Step 1. We borrow some ideas from [29] to prove that $t_\varepsilon, s_\varepsilon$ are bounded for $\varepsilon > 0$ sufficiently small. We proceed by contradiction and distinguish between two cases.

Case I. Both $s_\varepsilon, t_\varepsilon$ are unbounded for ε small. If $|t_\varepsilon|/s_\varepsilon \rightarrow \infty$, as $\varepsilon \rightarrow 0$, then

$$\begin{aligned} c_\varepsilon &\leq \Phi_\varepsilon(s_\varepsilon z + t_\varepsilon(\varphi_\varepsilon, -\varphi_\varepsilon)) \\ &= s_\varepsilon^2 \|z\|_\varepsilon^2 - t_\varepsilon^2 + t_\varepsilon s_\varepsilon O(1) - \int_{\mathbb{R}^2} F(s_\varepsilon u + t_\varepsilon \varphi_\varepsilon) + G(s_\varepsilon v - t_\varepsilon \varphi_\varepsilon) \\ &\leq s_\varepsilon^2 \|z\|_\varepsilon^2 - t_\varepsilon^2 + t_\varepsilon s_\varepsilon O(1) = s_\varepsilon^2 (O(1) - 1) \rightarrow -\infty, \end{aligned}$$

which contradict the fact $c_\varepsilon \geq c_0 > 0$. If $|t_\varepsilon|/s_\varepsilon \rightarrow 0$, as $\varepsilon \rightarrow 0$, then

$$\begin{aligned} c_\varepsilon &\leq s_\varepsilon^2 \|z\|_\varepsilon^2 - t_\varepsilon^2 + t_\varepsilon s_\varepsilon O(1) - \int_{\mathbb{R}^2} F(s_\varepsilon u + t_\varepsilon \varphi_\varepsilon) + G(s_\varepsilon v - t_\varepsilon \varphi_\varepsilon) \\ &= s_\varepsilon^3 \left(o(1) - \int_{\mathbb{R}^2} \frac{F(s_\varepsilon u + t_\varepsilon \varphi_\varepsilon)}{s_\varepsilon^3} + \frac{G(s_\varepsilon v - t_\varepsilon \varphi_\varepsilon)}{s_\varepsilon^3} \right). \end{aligned}$$

Since $c_\varepsilon \geq c_0 > 0$, as $\varepsilon \rightarrow 0$ we have

$$\int_{\mathbb{R}^2} \frac{F(s_\varepsilon u + t_\varepsilon \varphi_\varepsilon)}{s_\varepsilon^3} \rightarrow 0, \quad \frac{G(s_\varepsilon v - t_\varepsilon \varphi_\varepsilon)}{s_\varepsilon^3} \rightarrow 0.$$

Recalling that f has Moser critical growth at infinity, there exists $C > 0$ such that $|F(t)| \geq C|t|^3$ for $|t| \geq 1$. Let $A_\varepsilon := \{x \in \mathbb{R}^2 : |s_\varepsilon u(x) + t_\varepsilon \varphi_\varepsilon(x)| \geq 1\}$, then

$$\int_{A_\varepsilon} \frac{F(s_\varepsilon u + t_\varepsilon \varphi_\varepsilon)}{s_\varepsilon^3} \geq C \int_{A_\varepsilon} \left| u(x) + \frac{t_\varepsilon}{s_\varepsilon} \varphi_\varepsilon(x) \right|^3,$$

where the left hand side vanishes as $k \rightarrow \infty$, which yields $\lim_{\varepsilon \rightarrow 0} \int_{A_\varepsilon} |u(x)|^3 = 0$. At the same time,

$$\begin{aligned} \int_{\mathbb{R}^2 \setminus A_\varepsilon} |u(x)|^3 &\leq \int_{\mathbb{R}^2 \setminus A_\varepsilon} u^2(x) \left(\frac{1}{s_\varepsilon} + \frac{|t_\varepsilon|}{s_\varepsilon} |\varphi_\varepsilon| \right) \\ &\leq \frac{1}{s_\varepsilon} \int_{\mathbb{R}^2} u^2(x) + \frac{|t_\varepsilon|}{s_\varepsilon} \left(\int_{\mathbb{R}^2} u^4(x) \right)^{1/2} \left(\int_{\mathbb{R}^2} \varphi_\varepsilon^2(x) \right)^{1/2} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

Hence $\int_{\mathbb{R}^2} |u|^3 = 0$ and in turn $u \equiv 0$. Similarly, $v \equiv 0$. So that we get $c_* = 0$, which is a contradiction. If $|t_\varepsilon|/s_\varepsilon \rightarrow l > 0$, as $\varepsilon \rightarrow 0$, then following the same line as above,

$$\int_{A_\varepsilon} \left| u(x) + \frac{t_\varepsilon}{s_\varepsilon} \varphi_\varepsilon(x) \right|^3 \rightarrow 0.$$

Moreover,

$$\int_{\mathbb{R}^2 \setminus A_\varepsilon} \left| u(x) + \frac{t_\varepsilon}{s_\varepsilon} \varphi_\varepsilon(x) \right|^3 \leq \frac{1}{s_\varepsilon} \int_{\mathbb{R}^2 \setminus A_\varepsilon} \left| u(x) + \frac{t_\varepsilon}{s_\varepsilon} \varphi_\varepsilon(x) \right|^2 \rightarrow 0, \text{ as } \varepsilon \rightarrow 0.$$

Then

$$\int_{\mathbb{R}^2} \left| u(x) + \frac{t_\varepsilon}{s_\varepsilon} \varphi_\varepsilon(x) \right|^3 \rightarrow 0, \quad \varepsilon \rightarrow 0$$

and analogously

$$\int_{\mathbb{R}^2} \left| v(x) - \frac{t_\varepsilon}{s_\varepsilon} \varphi_\varepsilon(x) \right|^3 \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

So we get $\int_{\mathbb{R}^2} |u+v|^3 = 0$, that is $u = -v$. This implies $z = (u, v) \in E^-$ which contradicts the fact $z \in \mathcal{N}$.

Case II. Just one between s_ε and t_ε stays bounded for ε small. If $|t_\varepsilon|/s_\varepsilon \rightarrow \infty$, as $\varepsilon \rightarrow 0$, then $|t_\varepsilon| \rightarrow \infty$, as $\varepsilon \rightarrow 0$ and as above one has

$$c_\varepsilon \leq s_\varepsilon^2 \|z\|_\varepsilon^2 - t_\varepsilon^2 + t_\varepsilon s_\varepsilon O(1) = t_\varepsilon^2 (O(1) - 1) \rightarrow -\infty,$$

which contradicts the fact $c_\varepsilon \geq c_0 > 0$. If $|t_\varepsilon|/s_\varepsilon$ is bounded for ε small, then $s_\varepsilon \rightarrow \infty$ and $|t_\varepsilon|/s_\varepsilon \rightarrow 0$, as $\varepsilon \rightarrow 0$. Reasoning as in *Case I*, we get $u = v = 0$ and $c_* = 0$, which is again a contradiction.

Step 2. Recall that

$$c_\varepsilon \leq \max_{\omega \in \hat{E}_\varepsilon(z)} \Phi_\varepsilon(\omega) = \Phi_\varepsilon(\hat{m}_\varepsilon(z))$$

where $\hat{m}_\varepsilon(z) = s_\varepsilon z + t_\varepsilon(\varphi_\varepsilon, -\varphi_\varepsilon)$. Then

$$\begin{aligned} c_\varepsilon &\leq \Phi_\varepsilon(s_\varepsilon z + t_\varepsilon(\varphi_\varepsilon, -\varphi_\varepsilon)) = \Phi(s_\varepsilon z + t_\varepsilon(\varphi_\varepsilon, -\varphi_\varepsilon)) \\ &\quad + \int_{\mathbb{R}^2} (V_\varepsilon(x) - 1)(s_\varepsilon u + t_\varepsilon \varphi_\varepsilon)(s_\varepsilon v - t_\varepsilon \varphi_\varepsilon) \\ &\leq \max_{\omega \in \hat{E}(z)} \Phi(\omega) + I_\varepsilon = c_* + I_\varepsilon, \end{aligned}$$

where $I_\varepsilon := \int_{\mathbb{R}^2} (V_\varepsilon(x) - 1)(s_\varepsilon u + t_\varepsilon \varphi_\varepsilon)(s_\varepsilon v - t_\varepsilon \varphi_\varepsilon)$. Since $0 \in \mathcal{M}$, by Lebesgue's dominated convergence theorem and Step 1, we have

$$\begin{aligned} I_\varepsilon &= \int_{\mathbb{R}^2} (V_\varepsilon(x) - 1)[s_\varepsilon^2 uv - t_\varepsilon^2 \varphi_\varepsilon^2 + t_\varepsilon s_\varepsilon (v - u) \varphi_\varepsilon] \\ &\leq \int_{\mathbb{R}^2} (V_\varepsilon(x) - 1)[s_\varepsilon^2 uv + t_\varepsilon s_\varepsilon (v - u) \varphi_\varepsilon] \\ &\leq s_\varepsilon^2 \int_{\mathbb{R}^2} (V_\varepsilon(x) - 1) uv + |t_\varepsilon s_\varepsilon| \left(\int_{\mathbb{R}^2} |V_\varepsilon(x) - 1|^2 (v - u)^2 \right)^{1/2} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

Therefore, $\limsup_{\varepsilon \rightarrow 0} c_\varepsilon \leq c_*$. \square

3.5. Existence of ground state solutions for (5.1). For any $\lambda > 0$, let us consider the following problem in \mathbb{R}^2

$$(5.11) \quad \begin{cases} -\Delta u + \lambda u = g(v) \\ -\Delta v + \lambda v = f(u) \end{cases}$$

whose corresponding energy functional is

$$\Phi_\lambda(z) := \int_{\mathbb{R}^2} \nabla u \nabla v + \lambda uv - I(z), \quad z = (u, v) \in E.$$

As above one can define the generalized Nehari Manifold \mathcal{N}_λ and the least energy

$$c_\lambda := \inf_{z \in \mathcal{N}_\lambda} \Phi_\lambda(z).$$

Moreover, with the same assumptions of Theorem 2.1, if $c_\lambda \in (0, 4\pi/\alpha_0)$ for some $\lambda > 0$, then there exists $z_\lambda = (u_\lambda, v_\lambda) \in \mathcal{N}_\lambda$ such that $\Phi_\lambda(z_\lambda) = c_\lambda$.

Lemma 3.8. *With the assumptions of Theorem 2.1, for any $\lambda > 0$ the map $\lambda \mapsto c_\lambda \in (0, 4\pi/\alpha_0)$ is strictly increasing.*

Proof. For any $\lambda > 0$ with $c_\lambda \in (0, 4\pi/\alpha_0)$, let $z_\lambda = (u_\lambda, v_\lambda)$ be a solution of (5.11), then $\tilde{z}_\lambda = (\tilde{u}_\lambda, \tilde{v}_\lambda) = (u_\lambda(\cdot/\sqrt{\lambda}), v_\lambda(\cdot/\sqrt{\lambda}))$ satisfies in the whole plane the following system

$$(5.12) \quad \begin{cases} -\Delta \tilde{u}_\lambda + \tilde{u}_\lambda = \lambda^{-1} g(\tilde{v}_\lambda) \\ -\Delta \tilde{v}_\lambda + \tilde{v}_\lambda = \lambda^{-1} f(\tilde{u}_\lambda) \end{cases}$$

whose corresponding energy functional is

$$\tilde{\Phi}_\lambda(\tilde{z}_\lambda) := \int_{\mathbb{R}^2} \nabla \tilde{u}_\lambda \nabla \tilde{v}_\lambda + \tilde{u}_\lambda \tilde{v}_\lambda - \lambda^{-1} I(\tilde{z}_\lambda).$$

Similar as above, we can define the generalized Nehari Manifold $\tilde{\mathcal{N}}_\lambda$ and the least energy

$$\tilde{c}_\lambda := \inf_{z \in \tilde{\mathcal{N}}_\lambda} \tilde{\Phi}_\lambda(z).$$

We have $c_\lambda = \tilde{c}_\lambda \in (0, 4\pi/\alpha_0)$. Then (5.12) admits a ground state solution $\tilde{z}_\lambda = (\tilde{u}_\lambda, \tilde{v}_\lambda)$. Moreover,

$$\tilde{c}_\lambda := \inf_{z \in E \setminus E^-} \max_{\omega \in \hat{E}(z)} \tilde{\Phi}_\lambda(\omega) = \max_{\omega \in \hat{E}(\tilde{z}_\lambda)} \tilde{\Phi}_\lambda(\omega).$$

To show that c_λ is strictly increasing, it is enough to prove that \tilde{c}_λ is strictly increasing. For any $0 < \mu < \lambda$, the set $\hat{E}(\tilde{z}_\lambda)$ intersects $\tilde{\mathcal{N}}_\mu$ at exactly one point $\hat{m}_\mu(z)$, which is the unique global maximum point of $\tilde{\Phi}_\mu|_{\hat{E}(\tilde{z}_\lambda)}$. Since $F(s), G(s) > 0$ for any $s \neq 0$,

$$\begin{aligned} \tilde{c}_\mu &\leq \max_{\omega \in \hat{E}(\tilde{z}_\lambda)} \tilde{\Phi}_\mu(\omega) = \tilde{\Phi}_\mu(\hat{m}_\mu(z)) \\ &< \tilde{\Phi}_\lambda(\hat{m}_\mu(z)) \leq \max_{\omega \in \hat{E}(\tilde{z}_\lambda)} \tilde{\Phi}_\lambda(\omega) = \tilde{c}_\lambda. \end{aligned}$$

Therefore, $c_\mu < c_\lambda$. \square

Now, we are set to prove that the weak limit obtained in Proposition 3.6 is non trivial, precisely

Lemma 3.9. $z_\varepsilon \not\equiv 0$ provided $\varepsilon > 0$ is sufficiently small.

Proof. Assume by contradiction that $z_\varepsilon = 0$ for $\varepsilon > 0$ small, then $z_n = (u_n, v_n) \rightharpoonup 0$ in E_ε and $z_n \xrightarrow{a.e.} 0$ in \mathbb{R}^2 , as $n \rightarrow \infty$. It is well known that $\{z_n\}$ satisfies just one of the following alternatives:

1) (Vanishing)

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^2} \int_{B_R(y)} (u_n^2 + v_n^2) dx = 0 \quad \text{for all } R > 0;$$

2) (Nonvanishing) there exist $\nu > 0$, $R_0 > 0$ and $\{y_n\} \subset \mathbb{R}^2$ such that

$$\lim_{n \rightarrow \infty} \int_{B_{R_0}(y_n)} (u_n^2 + v_n^2) dx \geq \nu.$$

Due to $c_\varepsilon \in (c_0, 4\pi/\alpha_0)$ we can rule out *Vanishing*. So that *Nonvanishing* occurs. Let $\tilde{u}_n(\cdot) := u_n(\cdot + y_n)$ and $\tilde{v}_n(\cdot) := v_n(\cdot + y_n)$, then $|y_n| \rightarrow \infty$, as $n \rightarrow \infty$ and

$$(5.13) \quad \lim_{n \rightarrow \infty} \int_{B_{R_0}(0)} (\tilde{u}_n^2 + \tilde{v}_n^2) dx \geq \nu.$$

Let $\tilde{z}_n = (\tilde{u}_n, \tilde{v}_n)$, $\{\tilde{z}_n\}$ is bounded in E . Up to a subsequence, by (5.13) we assume that $\tilde{z}_n \rightarrow \tilde{z} \neq 0$ weakly in E for some $\tilde{z} = (\tilde{u}, \tilde{v}) \in E$ and $\Phi'_{V_\infty}(\tilde{z}) = 0$, where

$$\Phi_{V_\infty}(z) = \int_{\mathbb{R}^2} \nabla u \nabla v + V_\infty uv - I(z), \quad z = (u, v) \in E.$$

By (H2) and Fatou's Lemma, for fixed $\varepsilon > 0$,

$$\begin{aligned} c_\varepsilon + o_n(1) &= \Phi_\varepsilon(\tilde{z}_n) - \frac{1}{2} \langle \Phi'_\varepsilon(\tilde{z}_n), \tilde{z}_n \rangle \\ &= \int_{\mathbb{R}^2} \frac{1}{2} f(\tilde{u}_n) \tilde{u}_n - F(\tilde{u}_n) + \int_{\mathbb{R}^2} \frac{1}{2} g(\tilde{v}_n) \tilde{v}_n - G(\tilde{v}_n) \\ &\geq \int_{\mathbb{R}^2} \frac{1}{2} f(\tilde{u}) \tilde{u} - F(\tilde{u}) + \int_{\mathbb{R}^2} \frac{1}{2} g(\tilde{v}) \tilde{v} - G(\tilde{v}) + o_n(1) \\ &= \Phi_{V_\infty}(\tilde{z}) - \frac{1}{2} \langle \Phi'_{V_\infty}(\tilde{z}), \tilde{z} \rangle + o_n(1) \geq c_{V_\infty} + o_n(1). \end{aligned}$$

It follows that $c_\varepsilon \geq c_{V_\infty}$ for $\varepsilon > 0$ small enough. By Lemma 3.7 and Lemma 3.8, we get $c_{V_\infty} > c_*$. Again by Lemma 3.7 we get a contradiction. \square

By virtue of Lemma 3.9 we get straightforward the following

Corollary 3.10. For $\varepsilon > 0$ small enough, $\Phi_\varepsilon(z_\varepsilon) = c_\varepsilon$, namely z_ε is a ground state solution of (5.1).

3.6. Concentration. Reasoning as in Proposition 2.7 we have

Proposition 3.11. *Let $\varepsilon > 0$ and $z_\varepsilon = (u_\varepsilon, v_\varepsilon)$ be a ground state solution to (5.1). Then, $u_\varepsilon, v_\varepsilon \in L^\infty(\mathbb{R}^2) \cap C_{loc}^{1,\gamma}(\mathbb{R}^2)$ for some $\gamma \in (0, 1)$. Moreover, $u_\varepsilon(x), v_\varepsilon(x) \rightarrow 0$, as $|x| \rightarrow \infty$.*

By Proposition 3.11, there exists $y_\varepsilon \in \mathbb{R}^2$ such that

$$|u_\varepsilon(y_\varepsilon)| + |v_\varepsilon(y_\varepsilon)| = \max_{x \in \mathbb{R}^2} (|u_\varepsilon(x)| + |v_\varepsilon(x)|).$$

Moreover, $x_\varepsilon := \varepsilon y_\varepsilon$ is a maximum point of $|\varphi_\varepsilon(x)| + |\psi_\varepsilon(x)|$, where $(\varphi_\varepsilon(\cdot), \psi_\varepsilon(\cdot)) = (u_\varepsilon(\cdot/\varepsilon), v_\varepsilon(\cdot/\varepsilon))$ is a ground state solution of the original problem (1.1). We conclude the proof of Theorem 1.2 by proving Proposition 3.12, 3.13 and 3.14 below.

Proposition 3.12.

- 1) $\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon, \mathcal{M}) = 0$;
- 2) $(u_\varepsilon(\cdot + x_\varepsilon/\varepsilon), v_\varepsilon(\cdot + x_\varepsilon/\varepsilon))$ converges (up to a subsequence) to a ground state solution of

$$(5.14) \quad \begin{cases} -\Delta u + V_0 u = g(v) \\ -\Delta v + V_0 v = f(u) \end{cases} \quad \text{in } \mathbb{R}^2$$

- 3) $u_\varepsilon(x + x_\varepsilon/\varepsilon), v_\varepsilon(x + x_\varepsilon/\varepsilon) \rightarrow 0$, uniformly as $|x| \rightarrow \infty$, for $\varepsilon > 0$ sufficiently small.

Proof. By virtue of Proposition 3.5 and Fatou's Lemma, there exists $C > 0$ (independent of ε) such that $\|(u_\varepsilon, v_\varepsilon)\|_\varepsilon \leq C$ for all $\varepsilon \in (0, \varepsilon_0)$. Up to a subsequence, we may assume $z_\varepsilon = (u_\varepsilon, v_\varepsilon) \rightharpoonup z_0 = (u_0, v_0)$ in E and $(u_\varepsilon, v_\varepsilon) \xrightarrow{a.e.} (u_0, v_0)$ in \mathbb{R}^2 , as $\varepsilon \rightarrow 0$. Due to $c_\varepsilon \in (c_0, 4\pi/\alpha_0)$ for $\varepsilon > 0$ sufficiently small, as in Lemma 3.9, we have $u_0 \not\equiv 0, v_0 \not\equiv 0$. Moreover, $\Phi'(z_0) = 0$. By (H2) and Fatou's Lemma,

$$\begin{aligned} c_\varepsilon &= \Phi_\varepsilon(z_\varepsilon) - \frac{1}{2} \langle \Phi'_\varepsilon(z_\varepsilon), z_\varepsilon \rangle \\ &= \int_{\mathbb{R}^2} \frac{1}{2} f(u_\varepsilon) u_\varepsilon - F(u_\varepsilon) + \int_{\mathbb{R}^2} \frac{1}{2} g(v_\varepsilon) v_\varepsilon - G(v_\varepsilon) \\ &\geq \int_{\mathbb{R}^2} \frac{1}{2} f(u_0) u_0 - F(u_0) + \int_{\mathbb{R}^2} \frac{1}{2} g(v_0) v_0 - G(v_0) + o_\varepsilon(1) \\ &= \Phi(z_0) - \frac{1}{2} \langle \Phi'(z_0), z_0 \rangle + o_\varepsilon(1) \geq c_* + o_\varepsilon(1). \end{aligned}$$

Thanks to Lemma 3.7, $\Phi(z_0) = c_*$, namely (u_0, v_0) is a ground state solution of (5.14). Thanks to Fatou's Lemma again,

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{1}{2} f(u_\varepsilon) u_\varepsilon - F(u_\varepsilon) = \int_{\mathbb{R}^2} \frac{1}{2} f(u_0) u_0 - F(u_0)$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{1}{2} g(v_\varepsilon) v_\varepsilon - G(v_\varepsilon) = \int_{\mathbb{R}^2} \frac{1}{2} g(v_0) v_0 - G(v_0).$$

Repeating the argument in Proposition 2.6, we get $\|u_\varepsilon\|_\varepsilon \rightarrow \|u_0\|_{H^1}$ and $\|v_\varepsilon\|_\varepsilon \rightarrow \|v_0\|_{H^1}$, as $\varepsilon \rightarrow 0$. This implies $(u_\varepsilon, v_\varepsilon) \rightarrow (u_0, v_0)$ strongly in E as $\varepsilon \rightarrow 0$. Then, as in Proposition 2.8 and 2.9, $\{\|u_\varepsilon\|_\infty, \|v_\varepsilon\|_\infty\}$ is uniformly bounded for $\varepsilon > 0$ small and

$$\liminf_{\varepsilon \rightarrow 0} \min\{\|u_\varepsilon\|_\infty, \|v_\varepsilon\|_\infty\} > 0.$$

As in Proposition 2.11, there exists $R_2 > 0$ such that

$$\lim_{\varepsilon \rightarrow 0} \int_{B_{R_2}(x_\varepsilon/\varepsilon)} (u_\varepsilon^2 + v_\varepsilon^2) dx > 0.$$

Now, we claim that $\{x_\varepsilon\}$ is bounded for $\varepsilon > 0$ small enough. Suppose this does not occur, so that $|x_\varepsilon| \rightarrow \infty$, as $\varepsilon \rightarrow 0$. Let $\bar{u}_\varepsilon(\cdot) = u_\varepsilon(\cdot + x_\varepsilon/\varepsilon)$ and $\bar{v}_\varepsilon(\cdot) = v_\varepsilon(\cdot + x_\varepsilon/\varepsilon)$ which, up to a subsequence, $(\bar{u}_\varepsilon, \bar{v}_\varepsilon) \rightarrow \bar{z} = (\bar{u}, \bar{v})$ weakly in E , as $\varepsilon \rightarrow 0$ and $\bar{u}, \bar{v} \not\equiv 0$. Moreover, $\Phi'_{V_\infty}(\bar{z}) = 0$. As in Lemma 3.9 we get a contradiction. Therefore $\{x_\varepsilon\}$ is bounded for $\varepsilon > 0$ small. Up to a subsequence, assume $x_\varepsilon \rightarrow x_0$, as $\varepsilon \rightarrow 0$ and let $\hat{u}_\varepsilon(\cdot) = u_\varepsilon(\cdot + x_\varepsilon/\varepsilon)$, $\hat{v}_\varepsilon(\cdot) = v_\varepsilon(\cdot + x_\varepsilon/\varepsilon)$. Then, up to a subsequence, $\hat{z}_\varepsilon = (\hat{u}_\varepsilon, \hat{v}_\varepsilon) \rightarrow \hat{z} = (\hat{u}, \hat{v}) \neq 0$ weakly in E , as $\varepsilon \rightarrow 0$ and $\Phi'_{V(x_0)}(\hat{z}) = 0$, where

$$\Phi_{V(x_0)}(z) = \int_{\mathbb{R}^2} \nabla u \nabla v + V(x_0)uv - I(z), \quad z = (u, v) \in E.$$

By (H2) and Fatou's Lemma,

$$\begin{aligned} c_\varepsilon &= \Phi_\varepsilon(z_\varepsilon) - \frac{1}{2} \langle \Phi'_\varepsilon(z_\varepsilon), z_\varepsilon \rangle \\ &= \int_{\mathbb{R}^2} \frac{1}{2} f(\hat{u}_\varepsilon) \hat{u}_\varepsilon - F(\hat{u}_\varepsilon) + \int_{\mathbb{R}^2} \frac{1}{2} g(\hat{v}_\varepsilon) \hat{v}_\varepsilon - G(\hat{v}_\varepsilon) \\ &\geq \int_{\mathbb{R}^2} \frac{1}{2} f(\hat{u}) \hat{u} - F(\hat{u}) + \int_{\mathbb{R}^2} \frac{1}{2} g(\hat{v}) \hat{v} - G(\hat{v}) + o_\varepsilon(1) \\ &= \Phi_{V(x_0)}(\hat{z}) - \frac{1}{2} \langle \Phi'_{V(x_0)}(\hat{z}), \hat{z} \rangle + o_\varepsilon(1) \geq c_{V(x_0)} + o_\varepsilon(1). \end{aligned}$$

Recalling that $\limsup_{\varepsilon \rightarrow 0} c_\varepsilon \leq c_*$, we get $c_{V(x_0)} = c_*$ and hence (\hat{u}, \hat{v}) is a ground state solution of (5.14). Thanks to Lemma 3.8, $V(x_0) = V_0$, namely $x_0 \in \mathcal{M}$ and $\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon, \mathcal{M}) = 0$. Moreover, $(\hat{u}_\varepsilon, \hat{v}_\varepsilon) \rightarrow (\hat{u}, \hat{v})$ strongly in E , as $\varepsilon \rightarrow 0$. As in Proposition 2.8, $u_\varepsilon(x + x_\varepsilon/\varepsilon), v_\varepsilon(x + x_\varepsilon/\varepsilon) \rightarrow 0$ vanish at infinity uniformly in ε . \square

Proposition 3.13. *Let $(\varphi_\varepsilon, \psi_\varepsilon)$ be a ground state solution to (1.1) and $x_\varepsilon^1, x_\varepsilon^2$ be any maximum point of $|\varphi_\varepsilon|$ and $|\psi_\varepsilon|$ respectively. Then,*

$$\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon^i, \mathcal{M}) = 0, \quad \lim_{\varepsilon \rightarrow 0} |x_\varepsilon^i - x_\varepsilon| = 0, \quad i = 1, 2.$$

If in addition f and g are odd and (H6) holds, then for $\varepsilon > 0$ small enough, $\varphi_\varepsilon \psi_\varepsilon > 0$ in \mathbb{R}^2 and

$$\lim_{\varepsilon \rightarrow 0} |x_\varepsilon^1 - x_\varepsilon^2|/\varepsilon = 0.$$

Moreover, for some $c, C > 0$,

$$|\varphi_\varepsilon(x)| \leq C \exp(-\frac{c}{\varepsilon}|x - x_\varepsilon^1|), \quad |\psi_\varepsilon(x)| \leq C \exp(-\frac{c}{\varepsilon}|x - x_\varepsilon^2|), \quad x \in \mathbb{R}^2.$$

Proof. Note that $x_\varepsilon^1/\varepsilon, x_\varepsilon^2/\varepsilon$ are the maxima points of $u_\varepsilon, v_\varepsilon$ respectively. Thanks to the decay of $u_\varepsilon, v_\varepsilon$ and the following fact

$$\liminf_{\varepsilon \rightarrow 0} \min\{\|u_\varepsilon\|_\infty, \|v_\varepsilon\|_\infty\} > 0,$$

we get $|x_\varepsilon^i/\varepsilon - x_\varepsilon/\varepsilon|$ is bounded for $i = 1, 2$ and $\varepsilon > 0$ small enough. Then, $\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon^i, \mathcal{M}) = 0, i = 1, 2, \lim_{\varepsilon \rightarrow 0} |x_\varepsilon^i - x_\varepsilon| = 0, i = 1, 2$ and $\lim_{\varepsilon \rightarrow 0} |x_\varepsilon^1 - x_\varepsilon^2| = 0$. Next we assume f and g are odd, that (H6) holds, and also that, up to a subsequence, $(x_\varepsilon^1 - x_\varepsilon^2)/\varepsilon \rightarrow y_0 \in \mathbb{R}^2$, as $\varepsilon \rightarrow 0$. Let $\tilde{u}_\varepsilon(\cdot) = u_\varepsilon(\cdot + x_\varepsilon^1/\varepsilon)$ and $\tilde{v}_\varepsilon(\cdot) = v_\varepsilon(\cdot + x_\varepsilon^2/\varepsilon)$, then $(\tilde{u}_\varepsilon(\cdot), \tilde{v}_\varepsilon(\cdot + (x_\varepsilon^1 - x_\varepsilon^2)/\varepsilon)) \rightarrow (u, v) \neq 0$ strongly in E and in $C_{loc}^1(\mathbb{R}^2)$, as $\varepsilon \rightarrow 0$. Moreover, (u, v) is a ground state solution of (2.1). Without loss generality, we assume $u > 0, v > 0$ in \mathbb{R}^2 . Since 0 is a maximum point of \tilde{u}_ε , 0 is a maximum point also for u . By virtue of Theorem 2.2, 0 is the unique maximum point of u and v . On the other hand, up to a subsequence, $(\tilde{u}_\varepsilon(\cdot + (x_\varepsilon^2 - x_\varepsilon^1)/\varepsilon), \tilde{v}_\varepsilon(\cdot)) \rightarrow (\tilde{u}, \tilde{v}) \neq 0$ strongly in E and in $C_{loc}^1(\mathbb{R}^2)$, as $\varepsilon \rightarrow 0$. Then $(\tilde{u}(\cdot), \tilde{v}(\cdot)) = (u(\cdot - y_0), v(\cdot - y_0))$, which is a ground state solution of (2.1). Since 0 is a maximum point of \tilde{v}_ε , then 0 is the unique maximum point of \tilde{v} . Therefore, $y_0 = 0$.

Finally, we prove that $u_\varepsilon, v_\varepsilon$ do not change the sign for $\varepsilon > 0$ sufficiently small. Let

$$\bar{u}_\varepsilon = u_\varepsilon(\cdot + x_\varepsilon^1/\varepsilon), \quad \bar{v}_\varepsilon = v_\varepsilon(\cdot + x_\varepsilon^1/\varepsilon),$$

it is enough to prove $\bar{u}_\varepsilon \bar{v}_\varepsilon > 0$ in \mathbb{R}^2 . We assume $(\bar{u}_\varepsilon, \bar{v}_\varepsilon) \rightarrow (u, v) \in \mathcal{S}$ strongly in E and uniformly in $C_{loc}^2(\mathbb{R}^2)$, as $\varepsilon \rightarrow 0$ and 0 is the unique maximum point of u, v . By Theorem 2.2, $uv > 0$ in \mathbb{R}^2 . Without loss of generality, we assume $u > 0$ and $v > 0$ in \mathbb{R}^2 . Then there exist $R > 0$ and $\varepsilon_0 > 0$ such that $\bar{u}_\varepsilon, \bar{v}_\varepsilon > 0$ in $B_R(0)$ for $\varepsilon < \varepsilon_0$. Define

$$R_\varepsilon(\bar{u}_\varepsilon) := \sup\{r \mid \bar{u}_\varepsilon(x) > 0, \forall x \in B_r(0)\}, \quad R_\varepsilon(\bar{v}_\varepsilon) := \sup\{r \mid \bar{v}_\varepsilon(x) > 0, \forall x \in B_r(0)\}$$

and $R_\varepsilon := \min\{R_\varepsilon(\bar{u}_\varepsilon), R_\varepsilon(\bar{v}_\varepsilon)\}$, then $R_\varepsilon \geq R$ for any $\varepsilon < \varepsilon_0$. If $R_\varepsilon = \infty$ for any $\varepsilon < \varepsilon_0$, the proof is complete. Otherwise, there exists $\varepsilon_n > 0$ such that $\varepsilon_n \rightarrow 0$, as $n \rightarrow \infty$ and $R_n := R_{\varepsilon_n} < \infty$ for any fixed n . Then, by the maximum principle, $R_{\varepsilon_n}(\bar{u}_{\varepsilon_n}), R_{\varepsilon_n}(\bar{v}_{\varepsilon_n}) < \infty$ for any fixed $n \in \mathbb{N}$. Hence $\inf_{x \in \mathbb{R}^2} \bar{u}_{\varepsilon_n}(x) < 0$ and $\inf_{x \in \mathbb{R}^2} \bar{v}_{\varepsilon_n}(x) < 0$ for any $n \in \mathbb{N}$. Noting that $\bar{u}_{\varepsilon_n}(x), \bar{v}_{\varepsilon_n}(x) \rightarrow 0$, as $|x| \rightarrow \infty$, there exist $y_n, z_n \in \mathbb{R}^2$ such that $\bar{u}_{\varepsilon_n}(y_n) = \min_{x \in \mathbb{R}^2} \bar{u}_{\varepsilon_n}(x) < 0$ and $\bar{v}_{\varepsilon_n}(z_n) = \min_{x \in \mathbb{R}^2} \bar{v}_{\varepsilon_n}(x) < 0$. Then we have

$$g(\bar{v}_{\varepsilon_n}(y_n)) \leq V_0 \bar{u}_{\varepsilon_n}(y_n), \quad f(\bar{u}_{\varepsilon_n}(z_n)) \leq V_0 \bar{v}_{\varepsilon_n}(z_n).$$

By Remark 1.3 we have

$$V_0 \bar{u}_{\varepsilon_n}(y_n) \geq g(\bar{v}_{\varepsilon_n}(y_n)) \geq g(\bar{v}_{\varepsilon_n}(z_n)) \geq g\left(\frac{f(\bar{u}_{\varepsilon_n}(z_n))}{V_0}\right) \geq g\left(\frac{f(\bar{u}_{\varepsilon_n}(y_n))}{V_0}\right),$$

which yields $\inf_n |\bar{u}_{\varepsilon_n}(y_n)| > 0$ by (H1). Observe that $\bar{u}_{\varepsilon_n}(x) \rightarrow 0$, as $|x| \rightarrow \infty$, uniformly in ε , and thus $\sup_n |y_n| < \infty$, namely $|y_n| < R_n$ for n sufficiently large. Hence $\bar{u}_{\varepsilon_n}(y_n) > 0$, which is a contradiction. Finally, since $u_\varepsilon, v_\varepsilon$ do not change the sign, by the standard comparison principle, we get the uniformly exponential decay at infinity. \square

In order to complete the proof of Theorem 1.2 we need to prove the uniqueness of the maximum points of $\varphi_\varepsilon, \psi_\varepsilon$.

Proposition 3.14. *Let $x_\varepsilon^1, y_\varepsilon^1$ be any maxima points of φ_ε . Assume f and g are odd and (H6) holds. Then $x_\varepsilon^1 = y_\varepsilon^1$, for $\varepsilon > 0$ sufficiently small. Namely, the maximum point of φ_ε is unique. The same holds for ψ_ε .*

Proof. Let

$$\bar{u}_\varepsilon = u_\varepsilon(\cdot + x_\varepsilon^1/\varepsilon), \quad \bar{v}_\varepsilon = v_\varepsilon(\cdot + x_\varepsilon^1/\varepsilon).$$

Then $(\bar{u}_\varepsilon, \bar{v}_\varepsilon) \rightarrow (u, v) \in \mathcal{S}$ strongly in E and uniformly in $C_{loc}^2(\mathbb{R}^2)$, as $\varepsilon \rightarrow 0$. Moreover, there exist $c, C > 0$ such that

$$|\bar{u}_\varepsilon(x)| \leq C \exp(-c|x - x_\varepsilon^1/\varepsilon|), \quad x \in \mathbb{R}^2.$$

Hence $\|\bar{u}_\varepsilon\|_\infty \leq C \exp(-c|y_\varepsilon^1 - x_\varepsilon^1|/\varepsilon)$. As a consequence we have

$$\limsup_{\varepsilon \rightarrow 0} |y_\varepsilon^1 - x_\varepsilon^1|/\varepsilon < \infty.$$

Indeed, otherwise $\|\bar{u}_\varepsilon\|_\infty \rightarrow 0$, as $\varepsilon \rightarrow 0$, which yields

$$\int_{\mathbb{R}^2} [|\nabla \bar{v}_\varepsilon|^2 + V_\varepsilon(x + x_\varepsilon^1/\varepsilon)|\bar{v}_\varepsilon|^2] dx = \int_{\mathbb{R}^2} f(\bar{u}_\varepsilon)\bar{v}_\varepsilon dx \rightarrow 0.$$

Namely $\|v_\varepsilon\|_{1,\varepsilon} \rightarrow 0$, as $\varepsilon \rightarrow 0$ from which $\Phi_\varepsilon(u_\varepsilon, v_\varepsilon) \rightarrow 0$, as $\varepsilon \rightarrow 0$, thus a contradiction by Proposition 3.2. Therefore $|y_\varepsilon^1 - x_\varepsilon^1|/\varepsilon$ stays bounded for $\varepsilon > 0$ small. As in Proposition 3.13, $|y_\varepsilon^1 - x_\varepsilon^1|/\varepsilon \rightarrow 0$, as $\varepsilon \rightarrow 0$. Obverse that $\nabla \bar{u}_\varepsilon(0) = \nabla \bar{u}_\varepsilon((y_\varepsilon^1 - x_\varepsilon^1)/\varepsilon) = 0$. By Theorem 2.2, $\Delta u(0) < 0$. Recalling that $u(x) = u(|x|)$, $u'(0) = 0$ and $u''(r) < 0$ for $r = |x|$ small. On the other hand, since $g \in C^1$, $\bar{u}_\varepsilon \in C^2$ and $\bar{u}_\varepsilon \rightarrow u$ in $C_{loc}^2(\mathbb{R}^2)$, as $\varepsilon \rightarrow 0$, it follows from [24, Lemma 4.2] that $y_\varepsilon^1 = x_\varepsilon^1$ for $\varepsilon > 0$ sufficiently small. \square

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